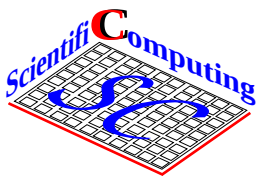

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Literature Review

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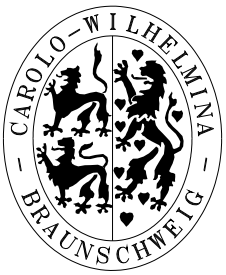

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Zusammenfassung

During the last years the interest in least squares finite element methods (LSFEM) has grown continuously. Least squares finite element methods offer some advantages over the widely used Galerkin variational principle. One reason is the ability to cope with first order differential operators without special treatment as required by the Galerkin FEM. The other reason comes from the numerical point of view, where the LSFEM leads to symmetric positive definite matrices which can be solved very efficiently under some conditions. This report gives an overview about the recent literature which appeared in the field of least squares finite element methods and summarises the essential results and facts about the LSFEM.

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1 Introduction

1.1 Development of the LSFEM

Inspired by the successful application of the Galerkin variational principle to structural problems, the basic idea was applied to other partial differential equations after 1965 ([108]). But numerical problems occurring in fluid and transport problems showed that the Galerkin method could exhibit numerical problems, if it is applied to non self adjoint partial differential equations (cf. section 9 in [108] and the references mentioned there). As a consequence, first works which casted arbitrary partial differential equations into an equivalent minimisation problem using a least squares principle in conjunction with the finite element ideas appeared also in that time ([75], [111]).

A huge part of the theoretical analysis of least squares methods is connected to the theory of elliptic systems. A major work in this area is a series of two papers from Agmon, Douglas and Nirenberg ([2],[3]), who developed the ADN-Theory which is used in many later publications about the least squares finite element method (LSFEM). Later on Wendland ([98]) used tools from complex function theory to establish several theorems for elliptic systems in two dimensional domains. One chapter is devoted to numerical methods for elliptic systems. As Wendland shows already, the straightforward application of a least squares principle may lead to suboptimal convergence. He points out that appropriate weights must be introduced to obtain optimal accuracy.

After that a more general theory especially for the LSFEM was developed by Aziz et al. in 1985 ([4]). They utilised the ADN-Theory to get a priori estimates for elliptic systems which then allow to prove optimal convergence rates (with respect to the used elements).

Beside these general theoretical works, several papers have been published which consider special problems and their treatment with the LSFEM.

In the beginning the focus of the research was on solving those problems with the LSFEM which are difficult to solve with the Galerkin finite element method. Today the focus has shifted to utilising the special properties of the least squares methods. One is the inherent ellipticity of the resulting formulations, which could lead to highly efficient multigrid algorithms if the correct first order formulation is used (cf. [53], [26], [72], [23]). The other special property is the built-in error estimator, which could be used to construct efficient adaptive algorithms (cf. [10], [77], [29]).

This report tries to summarise and present the current state of the research in the area of least squares finite element methods with focus on the first order methods. The next chapter will give a short overview over the theoretical basis of modern least squares methods and some related approaches. After that some important basic equations and their solution with the LSFEM will be shown. Two chapters about the LSFEM for the Stokes equations and the closely related equations of linear elasticity follow. The

Navier-Stokes equations, which extend the Stokes equations by a nonlinear convective term will be considered in the sixth chapter together with some theorems for the approximation of solutions to nonlinear partial differential equations. As this report cannot cover all areas of research in full length, the seventh chapter provides a short overview about some selected problems, which were successfully solved with the LSFEM. Next a short review of solution procedures proposed so far for the LSFEM is presented. This is quite important, because nice theoretical properties are useless without the ability to solve real world problems, which require efficient solution strategies. At the end of the report a short outlook about the future directions of the LSFEM will be given.

1.2 Notation

In this report the spatial domain of the problems considered will be called $\Omega \subset \mathbb{R}^d$, with $d = 1, 2, 3$ and assumed to have a sufficiently smooth boundary Γ. If different boundary conditions must be applied on different parts of the boundary, the parts of the boundary will be differentiated by a character index like Γ_n or Γ_d. Throughout this paper the standard notation will be used.

Derivatives of functions with multiple variables will be written in multi index notation:

$$D^\alpha = \frac{\partial^{\alpha_1} \dots \partial^{\alpha_d}}{\partial t_1^{\alpha_1} \dots \partial t_d^{\alpha_d}} \tag{1}$$

with $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbb{N}_0^d$. The space $C^k(\Omega)$ consists of functions which have continuous derivatives up to order k, that is $|\alpha| = \alpha_1 + \dots + \alpha_d \leq k$.

The space of square integrable functions defined on the domain Ω is denoted by L_2 and has the following scalar product:

$$\langle u, v \rangle_{0,\Omega} := \int_\Omega u \cdot v \, d\Omega \tag{2}$$

which induces the following norm:

$$\|u\|_{0,\Omega} := \langle u, u \rangle_{0,\Omega}^{1/2} = \left(\int_\Omega |u|^2 \, d\Omega \right)^{1/2}. \tag{3}$$

The following functions belong to $L_2(\Omega)$:

$$L_2(\Omega) := \{ u \,|\, \|u\|_{0,\Omega} < \infty \} \tag{4}$$

More generally the $L_p(\Omega)$ spaces are equipped with the norm

$$\|u\|_{0,p,\Omega} := \left(\int_\Omega |u|^p \, d\Omega \right)^{1/p} \tag{5}$$

and lead to the definition of the *Sobolev* spaces W_p^k

$$W_p^k(\Omega) := \{u \in L_p(\Omega) \,|\, \|u\|_{k,p,\Omega} < \infty\}. \tag{6}$$

with the norm

$$\|u\|_{k,p,\Omega} = \left(\sum_{|\alpha| \leq k} \|D_w^\alpha u\|_{0,p,\Omega}^p \right)^{1/p}. \tag{7}$$

Here D_w^α denotes the weak derivatives of order α. If the domain Ω is bounded and has a sufficiently smooth boundary, $H^k(\bar{\Omega}) = W_2^k(\Omega)$. A seminorm on H^k is defined by:

$$|u|_{k,\Omega} = \left(\sum_{|\alpha| = k} \|D_w^\alpha u\|_{0,2,\Omega}^2 \right)^{1/2}. \tag{8}$$

For some purposes also the following space is required:

$$H_0^k(\Omega) := \{u \in H^k(\Omega) \,|\, u(x) = 0, \quad \forall x \text{ on } \Gamma\} \tag{9}$$

In some parts of this report the following spaces will be used:

$$H(\mathrm{div}, \Omega)^n := \{\mathbf{u} \in (L_2(\Omega))^n \,|\, \nabla \cdot \mathbf{u} \in L_2(\Omega)\} \tag{10}$$

with the corresponding norm:

$$\|\mathbf{u}\|_{H(\mathrm{div},\Omega)} = (\|\mathbf{u}\|_{0,\Omega}^2 + \|\nabla \cdot \mathbf{u}\|_{0,\Omega}^2)^{1/2}. \tag{11}$$

The space H^{-1}, which is the dual space of H_0^1 will be equipped with the dual norm:

$$\|u\|_{-1} = \sup_{0 \neq q \in H_0^1} \frac{(u, q)}{\|q\|_1} \tag{12}$$

where $(\cdot, \cdot)$ denotes the duality mapping.

The equations will be written completely in terms of the Nabla-operator. This writing is unique except for the cross product, which leads to the *curl* operator and has two interpretations in two dimensions. If it is applied to a scalar, it should have the following meaning:

$$\nabla \times u = \begin{pmatrix} \frac{\partial u}{\partial y} \\ -\frac{\partial u}{\partial x} \end{pmatrix}. \tag{13}$$

Together with a vector $\mathbf{u} = (u_1, u_2)^T$ it is defined as follows:

$$\nabla \times \mathbf{u} = \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \tag{14}$$

For the displacement gradient formulations it will be convenient to use block vectors. If $\mathbf{u} = (u_1, u_2)$ is a vector, than an operator $\mathcal{G}$ defined on scalar functions will be extended component wise:

$$\mathbf{U} \equiv \mathcal{G}\mathbf{u} = \begin{pmatrix} \mathcal{G}u_1 \\ \mathcal{G}u_2 \end{pmatrix} \tag{15}$$

where $\mathbf{U}$ consists of the two vectors:

$$\mathbf{U} = \begin{pmatrix} \mathbf{u}_1 \\ \mathbf{u}_2 \end{pmatrix}. \tag{16}$$

An operator $\mathcal{D}$ on vector functions, will be extended component wise:

$$\mathcal{D}\mathbf{U} = \begin{pmatrix} \mathcal{D}\mathbf{u}_1 \\ \mathcal{D}\mathbf{u}_2 \end{pmatrix} \tag{17}$$

Unless convention dictates otherwise, scalar functions will be written with italic letters (u), vector valued functions will be denoted by bold letters ($\mathbf{u}$), tensors of second order or matrices will be denoted by capital bold letters ($\mathbf{M}$). Fourth order tensors will get the following typeset ($\mathbb{A}$), and material parameters will get small Greek letters (λ). Differential operators will be written in calligraphic letters ($\mathcal{L}$).

1.3 Introductory Example

To illustrate the basic ideas of the first order LSFEM, the comparatively simple instationary viscous Burger's equation will be used:

$$\frac{\partial u}{\partial t} - \epsilon \frac{\partial^2 u}{\partial x^2} + u \cdot \frac{\partial u}{\partial x} = f \quad \text{in} \quad [0..1] \tag{18}$$

$$u(x,0) = u_0 \tag{19}$$

$$u(0,t) = u(1,t) = 0 \tag{20}$$

where f will later be used to obtain an analytical solution, which allows a numerical error analysis.

1.3.1 First Order Formulation

As the name already implies, the first and most important step of the first order LSFEM the reformulation of the original problem into an equivalent first order system. Although this equation is pretty simple, several first order formulations can be found.

The first selection determines the meaning of the inevitable additional unknown. Either the flux $\partial u/\partial x$ or the negative flux $-\partial u/\partial x$ can be used which leads to one the following constraint equations:

$$\frac{\partial u}{\partial x} - p = 0 \tag{21}$$

$$\frac{\partial u}{\partial x} + p = 0. \tag{22}$$

Selecting Eq. (21), the second equation of the equivalent first order system becomes

$$\frac{\partial u}{\partial t} - \epsilon \frac{\partial p}{\partial x} + u \cdot \frac{\partial u}{\partial x} = f. \tag{23}$$

The derivation for the second variant is straightforward. Now it is still possible to replace the convective term $\partial u/\partial x$ in Eq. (23) by p, which would then lead to even more formulations.

Although all these variants are equivalent in the continuous case, the properties of the corresponding numerical schemes might differ. For sake of completeness it should be noted, that the equations could be multiplied by -1, which could also change some properties of the numerical methods.

1.3.2 Time Discretisation

Usually the time discretisation in the Galerkin method is done after the spatial discretisation. A spatial discretisation using the Galerkin principle transforms the instationary problem

$$\frac{\partial u}{\partial t} - \mathcal{L}u = f \tag{24}$$

into the following system of coupled ordinary differential equations

$$\frac{\partial u}{\partial t} + \mathbf{M}^{-1}\mathbf{K}u = \mathbf{M}^{-1} f \tag{25}$$

where $\mathbf{M}$ is the usual mass matrix and $\mathbf{K}$ represents the operator $\mathcal{L}$. This way cannot be used in the LSFEM (the details will be explained in subsection 3.3). Hence the temporal discretisation is usually performed before the spatial discretisation. Using the class of θ-methods for the time discretisation, Eqs. (21,23) become:

$$\frac{\partial u_{n+1}}{\partial x} - p_{n+1} = 0 \tag{26}$$

$$\begin{aligned} u_{n+1} + \Delta t \theta \left(-\epsilon \frac{\partial p_{n+1}}{\partial x} + u_{n+1} \cdot \frac{\partial u_{n+1}}{\partial x} \right) = \\ u_n + \Delta t \left(\theta f_{n+1} + (1-\theta) \left(f_n + \epsilon \frac{\partial p_n}{\partial x} - u_n \cdot \frac{\partial u_n}{\partial x} \right) \right). \end{aligned} \tag{27}$$

This strong form represents a slightly modified partial differential equation, which describes the solution at the next time step $n+1$ in terms of the solution at the previous time step n.

1.3.3 Linearisation

Now this nonlinear partial differential equation can directly be put into the least squares framework, which would then lead to a nonlinear functional, which has to be minimised. Another approach, which is suggested by Jiang [62] for the solution of the nonlinear Navier-Stokes equations is to linearise the strong form directly by using a Newton method or another suitable approach. According to [62] the advantage of the latter method is that the theory for the linear least squares FEM can directly be applied to the linear subproblems, which shifts the problems with the nonlinearity to the level of the strong form.

Hence we will follow that approach for this example. Each Newton iteration requires the solution of the following linear system of partial differential equations, where the index k denotes the iteration number:

$$\frac{\partial u_{n+1}^{k+1}}{\partial x} - p_{n+1}^{k+1} = 0 \tag{28}$$

$$u_{n+1}^{k+1} + \Delta t\theta \left(-\epsilon \frac{\partial p_{n+1}}{\partial x} + u_{n+1}^{k} \cdot \frac{\partial u_{n+1}^{k+1}}{\partial x} + u_{n+1}^{k+1} \cdot \frac{\partial u_{n+1}^{k}}{\partial x} \right) =$$
$$u_n + \Delta t \left(\theta f_{n+1} + (1-\theta) \left(f_n + \epsilon \frac{\partial p_n}{\partial x} - u_n \cdot \frac{\partial u_n}{\partial x} \right) \right) + \Delta t\theta u_{n+1}^{k} \cdot \frac{\partial u_{n+1}^{k}}{\partial x}. \tag{29}$$

For instationary problems, the solution of the previous time step generally seems to be a good initial guess.

1.3.4 Spatial Discretisation

Finally Eqs. (28–29) have to be discretised in space. For this purpose the least squares variational principle, which will b explained in more detail in section 2 is used. The starting point is a least squares functional, which consists of the sum of the squared norms of the equation residuals. Summarising the right hand side terms of Eq. (29)

into $\tilde{F}$, one possible functional based on squared L_2 norms is:

$$\mathcal{J}(u_{n+1}^{k+1}, p_{n+1}^{k+1}) = \left\| u_{n+1}^{k+1} + \Delta t\theta \left(-\epsilon \frac{\partial p_{n+1}}{\partial x} + u_{n+1}^{k} \cdot \frac{\partial u_{n+1}^{k+1}}{\partial x} + u_{n+1}^{k+1} \cdot \frac{\partial u_{n+1}^{k}}{\partial x} \right) - \tilde{F} \right\|_0^2 + \left\| \frac{\partial u_{n+1}^{k+1}}{\partial x} - p_{n+1}^{k+1} \right\|_0^2 \tag{30}$$

The exact solution of Eqs. (28–29) would lead to a functional value of zero. It is obvious, that an approximate solution would not let the value of the functional go to zero. But it is reasonable to assume that a better approximation would also reduce the value of the functional. Hence the essential idea of the least squares FEM is to minimise Eq. (30) over an approximation space. Details about the criteria which have to be satisfied to achieve convergence will be discussed in section 2.

At the minimum of the functional Eq. (30) the first variation will be zero, which leads to the variational formulation, which is used in the numerical method. Let $\mathbf{V}_h$ be a suitable subspace of the solution space $\mathbf{V}$. Then the least squares FEM seeks a $\mathbf{u}_h \in \mathbf{V}_h \subset \mathbf{V}$ such that

$$\mathcal{B}(\mathbf{u}_h, \mathbf{v}_h) = \mathcal{F}(\tilde{F}, \mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{V}_h \tag{31}$$

with $\mathbf{u}_h = (u_h, p_h)^T$ and $\mathbf{v}_h = (v_h, q_h)^T$. For this example:

$$\mathcal{B}(\mathbf{u}_h, \mathbf{v}_h) = \langle u_{n+1}^{k+1} + \Delta t\theta \left(-\epsilon \frac{\partial p_{n+1}}{\partial x} + u_{n+1}^{k} \cdot \frac{\partial u_{n+1}^{k+1}}{\partial x} + u_{n+1}^{k+1} \cdot \frac{\partial u_{n+1}^{k}}{\partial x} \right), v + \Delta t\theta \left(-\epsilon \frac{\partial q}{\partial x} + v \cdot \frac{\partial v}{\partial x} + v \cdot \frac{\partial v}{\partial x} \right) \rangle_{0,\Omega} + \left\langle \frac{\partial u_{n+1}^{k+1}}{\partial x} - p_{n+1}^{k+1}, \frac{\partial v}{\partial x} - q \right\rangle_{0,\Omega} \tag{32}$$

and

$$\mathcal{F}(\tilde{F}, \mathbf{v}_h) = \left\langle \tilde{F}, v + \Delta t\theta \left(-\epsilon \frac{\partial q}{\partial x} + v \cdot \frac{\partial v}{\partial x} + v \cdot \frac{\partial v}{\partial x} \right) \right\rangle_{0,\Omega} \tag{33}$$

1.3.5 Error Estimates

The mathematical analysis of the LSFEM for instationary problems is still in the beginning. Hence full convergence theorems, which make statements about the error between the exact space-time solution and the LSFEM solution are only available for a few

equations. Nevertheless, a necessary but not sufficient condition is the convergence of the solution for one timestep. If the error in one timestep cannot be controlled, it will not be possible to control the error over multiple timesteps.

Lax-Milgrams lemma guarantees the existence of a unique solution, if the bilinear form $\mathcal{B}$ is coercive (cf. definition 1) in some norm (cf. [84]). Due to the construction of the bilinear form, it will always be possible to show coercitivity in some weaker norms. If the finite element convergence rates should be utilised, it is necessary that the bilinear form is coercive in the correct spaces. For the standard finite element basis normally H^1-coercitivity is required. By using other finite element spaces, like Raviart-Thomas spaces which are a subspace of H_{div}, coercitivity in weaker norms like the H_{div} norm is sufficient.

For the bilinear form Eq. (32) the following estimates hold (could be found by a modification of the proof in section 3.4.2 of [62]):

$$c_1(\|u_h\|_1^2 + \|p_h\|_1^2) \leq \mathcal{B}(\mathbf{u}_h, \mathbf{u}_h) \leq c_2(\|u_h\|_1^2 + \|p_h\|_1^2). \tag{34}$$

This estimate is not very sharp, as the constants c_1 and c_2 depend on the time step size Δt.

Combining this with the usual finite element approximation properties (cf. subsection 2.2) the following error estimate can be found (the details can be found in section 2):

$$(\|u - u_h\|_1^2 + \|p - p_h\|_1^2) \leq c_3 h^{p-1}(\|u\|_1^2 + \|p\|_1^2) \tag{35}$$

which hold if $u \in H^1$ and $p \in H^1$. The index p denotes the polynomial degree of the used finite element space.

1.3.6 Numerical Verification

To verify the error estimate Eq. (35) some numerical tests were performed. As the purpose of this section was to introduce only the basic concepts, the error analysis will only consider the numerical error within one time step.

For this purpose the functions

$$u(x,t) = \sin(2\pi x) \tag{36}$$
$$p(x,t) = 2\pi \cos(2\pi x) \tag{37}$$

were used to determine the corresponding right hand side term f:

$$f = (4\pi^2 + 2\pi \cos(2\pi x)) \sin(2\pi x). \tag{38}$$

The initial conditions are prepared by interpolating the exact solution on the nodes.

h	$\|u-u_h\|_0$	$\|u-u_h\|_1$	$\|p-p_h\|_0$	$\|p-p_h\|_1$
2^{-3}	0.244559	1.972453	1.525837	15.483535
2^{-4}	0.134955	1.115410	0.877004	9.205664
2^{-5}	0.049778	0.451939	0.323723	3.696237
2^{-6}	0.014223	0.168518	0.091868	1.281603
2^{-7}	0.003696	0.070755	0.023779	0.485120
2^{-8}	0.000933	0.032947	0.005998	0.212854
2^{-9}	0.000234	0.016041	0.001503	0.101561
2^{-10}	0.000059	0.007937	0.000376	0.049969
2^{-11}	0.000015	0.003951	0.000094	0.024836
2^{-12}	0.000004	0.001971	0.000024	0.012387

Tabelle 1: Error after one time step of $\Delta t = 0.01$

The diffusion coefficient ϵ was set to 1 and the time step size was chosen to be $\Delta = 0.01$. Table 1 shows the errors in the L_2 and H^1 norm for different element sizes with linear elements.

The computational results confirm the error estimate Eq. (35). In the L_2 norm the error is reduced by a factor of 4 in each refinement. For the stronger H^1 norm the theory predicts a convergence rate of 2, which can be observed in Table 1.

2 Analysis of Least Squares Finite Element Methods

One reason for the success of finite element methods (FEM) is that it is possible to establish convergence theorems under reasonably general assumptions, while this is quite difficult for other numerical methods like the finite difference method (FDM). Also the LSFEM can be analysed with similar tools from functional analysis. In the next sections, a short overview about the general mathematical framework for the analysis of the LSFEM will be given. Starting with a description of the idea to cast partial differential equations into equivalent minimisation problems, the subsequent sections explain the importance of coercitivity in sufficiently strong norms to obtain an optimally accurate numerical method. After that some ideas from literature will be presented, which allow the derivation of the necessary coercitivity results for equations, where the standard approach based on L_2 norms does not succeed.

This introduction is mainly based on the following works [17], [62].

2.1 Basics

In the next sections we will consider the following system of linear partial differential equations including the special case of a single partial differential equation, which is denoted by the differential operator $\mathcal{L}$ in some domain Ω. The domain has a sufficiently smooth boundary and appropriate boundary conditions on the boundary of the domain Γ are prescribed by the operator $\mathcal{R}$:

$$\mathcal{L}\mathbf{u} = \mathbf{f} \quad \text{in} \quad \Omega \tag{39}$$

$$\mathcal{R}\mathbf{u} = \mathbf{g} \quad \text{on} \quad \Gamma \tag{40}$$

The general idea of the LSFEM is to seek a function $\mathbf{u} \in \mathbf{V}$, which minimises a functional $\mathcal{J}(\mathbf{u})$

$$\mathcal{J}(\mathbf{u}) = \frac{1}{2}\left(\|\mathcal{L}\mathbf{u} - \mathbf{f}\|^2_{\mathbf{X},\Omega} + \|\mathcal{R}\mathbf{u} - \mathbf{g}\|^2_{\mathbf{Y},\Gamma}\right) \tag{41}$$

which consists of the square of the residuum $\mathcal{L}\mathbf{u} - \mathbf{f}$ and $\mathcal{R}\mathbf{u} - \mathbf{g}$ in some norm $\|\cdot\|_{\mathbf{X},\Omega}$ and $\|\cdot\|_{\mathbf{Y},\Gamma}$ respectively. These norms should belong to Hilbert spaces $\mathbf{X}$ and $\mathbf{Y}$.

Usually for a single partial differential equation Sobolev spaces are used for $\mathbf{X}$ and $\mathbf{Y}$. For systems of partial differential equations, $\mathbf{X}$ and $\mathbf{Y}$ are normally products of different Sobolev spaces.

The boundary conditions can be treated in two different ways (cf. [88]). Either the functional Eq. (41) is directly used, or the boundary conditions are implemented by restricting the function space for the solution, in a way that the residual of Eq. (40) is always zero. In the following parts the latter variant will be assumed. Thus the elements of the function space $\mathbf{V}$ will satisfy Eq. (40).

To get a numerical method from this idea, the first variation of Eq. (41) has to be zero. This leads then to the following equation which is equivalent to Eq. (41):

$$\mathcal{B}(\mathbf{u},\mathbf{v}) = \mathcal{F}(\mathbf{f},\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{V} \tag{42}$$

with the following definitions

$$\mathcal{B}(\mathbf{u},\mathbf{v}) = \langle \mathcal{L}\mathbf{u}, \mathcal{L}\mathbf{v} \rangle_{\mathbf{X},\Omega}, \qquad \mathcal{F} = \langle \mathbf{f}, \mathcal{L}\mathbf{v} \rangle_{\mathbf{X},\Omega}. \tag{43}$$

The scalar product of $\mathbf{X}$ is denoted by $\langle \cdot,\cdot \rangle_{\mathbf{X}}$. Introducing discrete subspaces $\mathbf{V}_h$ of $\mathbf{V}$ gives then a method suited for the numerical solution of partial differential equations.

Although the basic principle can be applied to nearly arbitrary Hilbert spaces and differential operators, currently most research is focused on first order methods, which minimise the L_2 norm of a first order differential operator applied to the solution. Methods which use other norms, or allow higher order differential operators are difficult to use for practical applications (except the methods introduced in section 2.4).

In the normal finite element method the finite dimensional subspace $\mathbf{V}_h$ is constructed from piecewise Lagrange polynomials of arbitrary degree on some basic geometric domains (triangles, quadrilaterals, etc.) which build a triangulation of the domain Ω. If the interpolation functions on the small subdomains (the finite elements) are continuous over the element edges, $\mathbf{V}_h \subset C^0$. Therefore this space is often referred as C^0 finite element space. The first derivatives of this standard finite element space are elements of L_2 and can be measured in the L_2 norm. Thus a differential operator with higher derivatives would either require another norm or finite elements with a continuous first derivative (C^1 elements). These elements are impractical for most applications. Another disadvantage is the worse condition number which increases the numerical effort to find a solution of the resulting system of equations.

Some other finite element spaces, like the Raviart-Thomas spaces have been used in conjunction with the LSFEM as well ([29]). They are subspaces of the slightly more uncommon spaces like $H(\mathrm{div},\Omega)$, where only the divergence lies in L_2.

Using the L_2 space for $\mathbf{X}$ and assuming $\mathcal{L}$ to be of first order, the discrete version of Eq. (42) reads:

$$\langle \mathcal{L}\mathbf{u}_h, \mathcal{L}\mathbf{v}_h \rangle_{0,\Omega} = \langle \mathbf{f}, \mathcal{L}\mathbf{v}_h \rangle_{0,\Omega} \quad \forall \mathbf{v}_h \in \mathbf{V}_h \subset \mathbf{V}. \tag{44}$$

Due to the local support of normal finite elements, the discrete system can be assembled element by element (similar to the Galerkin FEM). It should be noted that the resulting system of linear equations is always symmetric positive definite because it stems from a minimisation problem.

2.2 Approximation Properties of Finite Elements

One part of the theoretical framework of the LSFEM are estimates which guarantee that the finite subspace $\mathbf{V}_h$ of the function space considered can approximate an element of the original space with arbitrary accuracy in some norm if the triangulation is made fine enough. Furthermore, these estimates give some information about the norm of the error, which normally depends on the typical element size of the triangulation and the polynomial degree of the shape functions on the element.

In the following lemmas, the triangulation $\mathcal{T}_h$ denotes a tessellation of the domain Ω with a polygonal boundary into triangles or quadrilaterals (or the corresponding elements in higher dimensions) κ such that:

$$\Omega = \bigcup_{\kappa \in \mathcal{T}_h} \kappa. \tag{45}$$

A general result which is true for such triangulations can be found in [55]:

Lemma 1 *Assume that Ω is a convex polygon. Let $\mathcal{T}_h$ be a regular triangulation of Ω. For real $p > 1$, the interpolation operator Π_h satisfies the following error estimate for all integers m and real r with $0 \leq m \leq r+1, 1 \leq r \leq k$:*

$$|v - \Pi_h v|_{m,p,\Omega} \leq c_1 h^{r+1-m} |v|_{r+1,p,\Omega} \quad \forall v \in W_p^{r+1}(\Omega) \tag{46}$$

and the constant $c_1 > 0$ is independent of h and v.

In this theorem, k denotes the polynomial degree of the shape functions on the elements in the triangulation $\mathcal{T}_h$ and h is the characteristic element size of the triangulation. The operator Π_h is an interpolation operator, which ensures that the value of the finite element function on the support points of the element (the nodes) is equal to the function value at those points (for details cf. [55]).

For the LSFEM especially the following two inverse estimates, which were taken from [55], are of great importance:

Lemma 2 *Let l and p be reals with $1 \leq l, p \leq \infty$. Under the assumption that the triangulation $\mathcal{T}_h$ is regular there exists a constant $c_1 > 0$ independent of h such that:*

$$|v|_{1,l,\Omega} \leq c_1 h^{-1+\min(0,d/l-d/p)} \|v\|_{0,p,\kappa} \quad \forall v \in \Theta_h. \tag{47}$$

Furthermore if m is a non-negative integer and $l \leq p$ or if the triangulation $\mathcal{T}_h$ is uniformly regular and $l > p$, there exists a constant c_2, independent of h, such that:

$$|v|_{m,l,\Omega} \leq c_2 h^{\min(0,d/l-d/p)} |v|_{m,p,\Omega} \quad \forall v \in \Theta_h. \tag{48}$$

The space Θ_h denotes the standard finite element space of piecewise polynomials and d should again denote the spatial dimension. Together with Lemma 1 this justifies the replacement of stronger norms through the weaker L_2 norm in the discrete case (cf. subsection 2.4.1).

2.3 Error Estimates

One desired property for numerical methods is obviously that they give optimal accuracy with respect to the numerical effort used to obtain a result. It is clear, that this is the case when the approximate solution found by the variational principle is the best which can be found in the discrete subspace. The next paragraphs show some fundamental results for the L_2 LSFEM with a small example, which can be found in [62].

First thing to show is, that the solution $\mathbf{u}_h$ found with the variational principle is the best minimiser of the residual norm. The variational formulation reads:

$$\begin{aligned} \langle \mathcal{L}\mathbf{u}, \mathcal{L}\mathbf{v} \rangle_0 &= \langle f, \mathcal{L}\mathbf{v} \rangle_0 \quad \forall \mathbf{v} \in \mathbf{V} && (49) \\ \langle \mathcal{L}\mathbf{u}_h, \mathcal{L}\mathbf{v}_h \rangle_0 &= \langle f, \mathcal{L}\mathbf{v}_h \rangle_0 \quad \forall \mathbf{v} \in \mathbf{V}_h && (50) \end{aligned}$$

and also because $\mathbf{V}_h \subset \mathbf{V}$

$$\langle \mathcal{L}\mathbf{u}, \mathcal{L}\mathbf{v}_h \rangle_0 = \langle f, \mathcal{L}\mathbf{v}_h \rangle_0 \quad \forall \mathbf{v} \in \mathbf{V}_h \tag{51}$$

Subtracting Eq. (50) from Eq. (51) gives:

$$\langle \mathcal{L}(\mathbf{u}_h - \mathbf{u}), \mathcal{L}\mathbf{v}_h \rangle_0 = 0 \quad \forall \mathbf{v}_h \in \mathbf{V}_h. \tag{52}$$

This equation is often called the orthogonality condition.

In the following part the finite element interpolation of $\mathbf{u}$ should be denoted by $\Pi_h \mathbf{u}$. Starting with

$$\|\mathcal{L}(\mathbf{u} - \mathbf{u}_h)\|_0^2 = \langle \mathcal{L}(\mathbf{u} - \mathbf{u}_h), \mathcal{L}(\mathbf{u} - \mathbf{u}_h) \rangle_0 \tag{53}$$

and adding ßero"we have:

$$\begin{aligned} \|\mathcal{L}(\mathbf{u} - \mathbf{u}_h)\|_0^2 = &\langle \mathcal{L}(\mathbf{u} - \mathbf{u}_h), \mathcal{L}(\mathbf{u} - \Pi_h \mathbf{u}) \rangle_0 + \\ &\langle \mathcal{L}(\mathbf{u} - \mathbf{u}_h), \mathcal{L}(\Pi_h \mathbf{u} - \mathbf{u}_h) \rangle_0 \end{aligned} \tag{54}$$

From Eq. (52) and because $\Pi_h \mathbf{u} - \mathbf{u}_h \in \mathbf{V}_h$:

$$\langle \mathcal{L}(\mathbf{u} - \mathbf{u}_h), \mathcal{L}(\Pi_h \mathbf{u} - \mathbf{u}_h) \rangle_0 = 0 \tag{55}$$

and hence:

$$\|\mathcal{L}(\mathbf{u} - \mathbf{u}_h)\|_0^2 = \langle \mathcal{L}(\mathbf{u} - \mathbf{u}_h), \mathcal{L}(\mathbf{u} - \Pi_h \mathbf{u}) \rangle_0 \tag{56}$$

Using the Schwarz inequality and dividing by $\|\mathcal{L}(\mathbf{u}-\mathbf{u}_h)\|_0$ we get the desired result:

$$\|\mathcal{L}(\mathbf{u}-\mathbf{u}_h)\|_0 \leq \|\mathcal{L}(\mathbf{u}-\Pi_h\mathbf{u})\|_0. \tag{57}$$

To find error estimates for the LSFEM it is necessary to make some assumptions about the solution and the bilinear form $\mathcal{B}$, which was found by applying the least squares principle to the linear differential operator $\mathcal{L}$.

Definition 1 *The bilinear form $\mathcal{B}(u,v)$ is said to be V-coercive if the following conditions are satisfied (cf. [84]):*

- $|\mathcal{B}(u,v)| \leq c_1\|u\|_V\|v\|_V \quad \forall u,v \in \boldsymbol{V}$
- $|\mathcal{B}(u,u)| \geq c_2\|u\|_V^2 \quad \forall u \in \boldsymbol{V}$

where c_1 and c_2 are independent of u and v.

Remark: In [84], a bilinear form satisfying this definition is called V-elliptic. Pedersen uses the term *coercive* for a definition which is named *weakly coercive* by other authors like in [85]. □

Continuity of the bilinear form, the first inequality in the definition, guarantees that a small change in the solution leads to a small change in the right hand side term of the equation. The second part is equivalent to the continuity of the inverse operator and guarantees that the solution will depend continuously on the right hand side term. By virtue of the Lax-Milgram lemma the V-coercitivity furthermore implies that the associated problem has a unique solution (cf. [55] and others).

A relatively weak assumption is the boundedness from below in a L_2 norm, which will always be satisfied by a L_2 based least squares method:

$$c_1\|\mathbf{u}\|_0 \leq \|\mathcal{L}\mathbf{u}\|_0. \tag{58}$$

Continuity can normally be shown in stronger norms without difficulties. Therefore we will assume that the following relation holds:

$$\|\mathcal{L}\mathbf{u}\|_0 \leq c_2\|\mathbf{u}\|_1. \tag{59}$$

Connecting these estimates with Eq. (57) the following inequality can be found:

$$c_1\|\mathbf{u}-\mathbf{u}_h\|_0 \leq \|\mathcal{L}(\mathbf{u}-\mathbf{u}_h)\|_0 \leq \|\mathcal{L}(\mathbf{u}-\Pi_h\mathbf{u})\|_0 \leq c_2\|\mathbf{u}-\Pi_h\mathbf{u}\|_1 \tag{60}$$

Dividing by c_2 and using Eq. (46) we get:

$$\frac{c_1}{c_2}\|\mathbf{u}-\mathbf{u}_h\|_0 \leq \|\mathbf{u}-\Pi_h\mathbf{u}\|_1 \leq c_3 h^r|\mathbf{u}|_{r+1,\Omega}. \tag{61}$$

This result has two implications. First it guarantees convergence of arbitrary first order differential operators under the assumptions in Eq. (58) and Eq. (59) at least in the L_2 norm, which can be seen as an advantage of the LSFEM. On the other hand the order of convergence is one order too low, which means the possible convergence rates of the elements are not completely utilised by the variational scheme. Furthermore no convergence in higher norms, like the H^1 norm is guaranteed.

If we assume $H^1 - coercitivity$, the following stronger relation holds:

$$c_4\|\mathbf{u}\|_1 \leq \|\mathcal{L}\mathbf{u}\|_0. \tag{62}$$

Now using the previous results it is easy to show that:

$$\frac{c_4}{c_5}\|\mathbf{u}-\mathbf{u}_h\|_1 \leq c_6 h^r |\mathbf{u}|_{r+1,\Omega}. \tag{63}$$

From this result L_2 convergence follows directly:

$$\frac{c_4}{c_5}\|\mathbf{u}-\mathbf{u}_h\|_0 \leq c_6 h^{r+1} |\mathbf{u}|_{r+1,\Omega}. \tag{64}$$

This time the convergence rates of the LSFEM are optimal with respect to the possibilities offered by the finite elements. Although this were pretty simple examples, they show very well the main difficulty of LSFEM methods. Thus most papers are concerned with estimates similar to Eq. (62) that guarantee optimal convergence rates.

2.4 Coercitivity and Stabilisation

The coercitivity of the bilinear form in sufficiently strong norms (normally H^1), which is necessary for optimal convergence rates, includes two closely connected subproblems. One is to prove the required estimates mathematically and the other is to find a suitable first order formulation for this purpose.

For the first subproblem two ways are used regularly in literature. One is limited to the elliptic systems of Agmon Douglis Nirenberg (ADN) type (cf. [2]) and was proposed generally for the LSFEM by Kellog and Aziz in [4]. In this approach the theory developed in [2] is utilised to establish some a priori regularity estimates, which then can be used to get the desired coercitivity estimates. Looking at a typical regularity result, which can be found with the ADN theory for a linear differential operator $\mathcal{L}$:

$$\|\mathbf{u}\|_q \leq c_1\|\mathbf{f}\|_{q-1} \tag{65}$$

and using the partial differential equation $\mathcal{L}\mathbf{u} = \mathbf{f}$ it is clear, that

$$\|\mathbf{u}\|_q^2 \leq c_1^2\|\mathcal{L}\mathbf{u}\|_{q-1}^2. \tag{66}$$

which implies the required coercitivity.

If the problem is neither elliptic nor of ADN type, the things become more complicated because the estimates have to be derived ”by handüsing theorems from functional analysis. A slight variation of this idea can be found in [62]. Jiang first derives the correct estimates including the required boundary conditions for the *div-curl* system and the *div-curl-grad* system. The *div-curl* system reads (cf. section 5.2 in [62]):

$$\begin{aligned} \nabla \times \mathbf{u} &= \omega \quad \text{in} \quad \Omega && (67)\\ \nabla \cdot \mathbf{u} &= \rho \quad \text{in} \quad \Omega && (68)\\ \mathbf{n} \cdot \mathbf{u} &= 0 \quad \text{on} \quad \Gamma_1 && (69)\\ \mathbf{n} \times \mathbf{u} &= 0 \quad \text{on} \quad \Gamma_2 && (70) \end{aligned}$$

For this system Jiang shows that

$$\|\mathbf{u}\|_1 \leq c_1(\|\omega\|_0 + \|\rho\|_0). \tag{71}$$

This estimate could easily be extended to the required coercitivity estimate and hence guarantees optimal convergence rates. The *div-curl-grad* system is:

$$\begin{aligned} \nabla \cdot \mathbf{p} &= -f \quad \text{in} \quad \Omega && (72)\\ \nabla \times \mathbf{p} &= 0 \quad \text{in} \quad \Omega && (73)\\ \nabla u - \mathbf{p} &= 0 \quad \text{in} \quad \Omega && (74)\\ u &= 0 \quad \text{on} \quad \Gamma && (75)\\ \mathbf{n} \times \mathbf{p} &= 0 \quad \text{on} \quad \Gamma && (76) \end{aligned}$$

With some specific modifications in the 2D and 3D version, Jiang shows again full H^1-coercitivity of the resulting least squares functional and thus also optimal convergence rates [62].

After that he shows that many important equations can be transformed into coupled *div-curl* and *div-curl-grad* systems. But the boundary conditions, which must be applied to achieve optimal convergence, are often not practical, which is a clear disadvantage of his proposed method.

Methods for the second subproblem can be interpreted differently and are accordingly also denoted differently in the literature. The goal is always to establish a coercitivity estimate for the bilinear form, which in conjunction with the Lax-Milgram Lemma guarantees the well posedness of the variational problem. This process can be seen and called as *stabilisation procedure* in compliance with the terms used in the normal Galerkin FEM. Similar to the normal FEM some problems do not require this stabilisation and allow the straightforward application of the LSFEM principle. But for other problems several ways have been proposed in literature:

- Weighted LSFEM
- H^{-1} LSFEM, Negative Norm LSFEM, Inverse Norm LSFEM
- Adding redundant terms, Augmented LSFEM
- Streamline Diffusion
- FOSLL*

2.4.1 Weighted LSFEM

For equations which consist of more than one part, the LSFEM minimises the sum of the residual norms of the different parts. This process can be seen as compromise solution between the different parts of the equations. Considering the Navier-Stokes equations as a small example, the LSFEM will violate the mass conservation a bit and the conservation of moment as well a little bit. Hence weighting the parts of the equation with factors can emphasise different properties depending on the objectives.

Weighting appears in literature in several forms with different justifications. Only the weighting to ïmitateä different norm comes from stabilisation but the other ideas related to weighting will be presented here as well to have the weighted LSFEM in a single section:

- To ïmitateä different norm
- Matrix weighting
- From physical arguments
- To reduce errors locally

For some problems the coercitivity cannot be shown in adequate norms if only L_2 norms are minimised. Replacing some of the L_2 norms with other norms like H^1 or H^{-1} the coercitivity can be established. The treatment of discrete negative norms is explained in subsection 2.4.2. But the direct evaluation of H^1 norms would again require C^1 ansatz-spaces.

Here weights offer a way to circumvent this difficulty. The essential argument is the fact that all norms on discrete subspaces, which are the ones actually used in computations, are equivalent up to a constant. It has been shown in [42] that the H^1 norm and the L_2 norm applied to a standard finite element function differ by a factor of h which characterises the element size:

$$c_1 h \|u_h\|_0 \leq \|u_h\|_1 \leq c_2 h \|u_h\|_0. \tag{77}$$

Thus the H^1 norm can often be replaced by a weighted L_2 norm. For other norms similar equivalence relations can be found. One drawback of these weighted methods is that the weighting influences the condition number of the corresponding discrete system and often no efficient solution method is known.

In [87] the weighting is not only applied to the different parts of the equation. Instead a complete symmetric positive definite weighting matrix is introduced, which allows a fine tuning of the LSFEM. The article considers a diffusion problem, which is transformed into the following first order system:

$$\mathcal{L}_1(p) := \frac{\partial p}{\partial x} \tag{78}$$

$$\mathcal{L}_2(u,p) := p + \frac{\partial u}{\partial x} \tag{79}$$

$$\tag{80}$$

The matrix weighted bilinear form, which is examined in [87], is then:

$$\mathcal{B}((u,p),(v,q)) = \frac{1}{2}\int_\Omega \begin{pmatrix} \mathcal{L}_1(q) \\ \mathcal{L}_2(v,q) \end{pmatrix}^T \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \begin{pmatrix} \mathcal{L}_1(p) \\ \mathcal{L}_2(u,p) \end{pmatrix} d\Omega \tag{81}$$

Clearly setting $\alpha_{11} = \alpha_{22} = 1$ would lead to the usual least squares formulation. Using modern symbolic mathematics software, optimal weighting parameters α_{ij} are derived from the analytical solution and it is shown that the correct weighting matrix can increase the accuracy in a test example significantly. But this technology seems to be in its infancies and due to its dependence on powerful symbolic mathematical software it might be impossible to find solutions for more complex equations. The situation is very similar to that encountered in stabilised methods, where the optimal choice of the stabilisation parameter is often not clear for more complex systems of equations (cf. [48]).

Another weighting is motivated by the physical quantities appearing in a system of coupled partial differential equations. While the classical FEM tries to satisfy all equations at least in average, the LSFEM only tries to minimise the sum of the residual norms. But the residual can have different physical units. To overcome these difficulties, some authors introduced scales to get a dimensionless form of the unknowns ([8],[50],[99], [7]).

A different intention for the use of weights can be found in the so called iteratively reweighted LSFEM, which was proposed in [61]. Jiang considered pure convection problems in two dimensions of the form

$$v_x(x,y)\frac{\partial u}{\partial x} + v_y(x,y)\frac{\partial u}{\partial y} = 0, \quad \text{on} \quad \Omega \tag{82}$$

with appropriate boundary conditions and a constant velocity field described by v_x and v_y. The pure LSFEM would minimise the residual norm of this equation and produce slightly diffusive results.

Because the normal LSFEM minimises the sum of the element wise L_2-residuals, each element contributes roughly the same relative part to the global residual. If the shock, which is the difficulty of the pure convection problems, runs through one element, the residual will naturally be relatively large in this element, because the finite elements cannot display shocks. Reducing the influence of this element onto the global residual would therefore lead to an improved quality of the overall solution. Jiangs idea is now to introduce element weights which are used to steer this wanted behaviour.

The complete algorithm consists of an iterative procedure. Starting with the normal LSFEM solution (i.e. equal weights for all elements) a shock detection operator runs over the solution and defines new element weights. Jiang proposes the following weights:

$$W_l = \frac{1}{|R_l|^6_{prev} + \epsilon} \tag{83}$$

$$W_l = \begin{cases} 10^{15} & \text{if} \quad |V|^6_{prev} \leq 10^{-7} \\ \frac{1}{|V|^6_{prev}} & \text{otherwise} \end{cases} \tag{84}$$

with R_l being the element L_2 residual of element l. V in the latter formula is the variation of the nodal values in one element defined by:

$$V = \sum_{m=1}^{N_{node}} |U_m - U_{m-1}| \qquad U_0 = U_{N_{node}} \tag{85}$$

The first weight function clearly detects high element residuals, which should indicate that the finite element approximation in that element is not sufficient. In the second proposed weight, strong gradients are used as an indicator that the shock runs through the element.

After evaluating the weights, a new solution is computed using the new weighting. A couple of iterations later, the weight on elements which are close to the shock is nearly zero and Jiang shows with some numerical examples that the width of the shock region is not larger than one element, which can be considered optimal for this kind of algorithm.

Some own numerical tests with this idea showed that the algorithm fails to converge in some cases. Hence the algorithm might not be well suited for real life applications due to the lack of robustness. Actually the proposed algorithm can be seen as a comparatively simple iterative scheme for the solution of a nonlinear system of equations. This perspective might explain the lack of robustness. As a consequence, using more

sophisticated iterative nonlinear schemes like Newton or Quasi-Newton methods could lead to a more robust scheme without changing the desired properties. To our knowledge this has not yet been done.

2.4.2 H^{-1} LSFEM

Negative norms have mainly two purposes in the LSFEM. For the Laplace operator the normal Galerkin approach allows the right hand side to be in H^{-1}. As the right hand side goes into the functional which is minimised, it has to be in L_2 for the normal LSFEM. Hence the regularity requirements are increased. Often this issue might be of minor importance but in some applications like mechanics the right hand side might not satisfy this requirement. Using a negative norm instead leads then to reduced regularity demands on the right hand side and the solution.

It was shown that the bilinear form stemming from the minimised functional must be coercive in some norm, which is related to the problem and the approximation space. For several first order formulations it is not possible get estimates like Eq. (62) by using only L_2 norms on the right hand side of the relation. Again negative norms can help to overcome the difficulties (An example can be found in section 4).

This norm has an awkward definition, but Glowinski et.al. discovered a way to handle it at least in the discrete case (cf. [56]). Later their idea was applied to the LSFEM ([20]) and since then has been used in several applications ([33], [11]). These methods are normally called after this norm H^{-1} *methods* or *negative norm methods*.

Looking at the definition of the norm in Eq. (12) it is easy to see that this norm cannot be evaluated directly, like the L_2 norm or the $H^k, k > 0$ norms. But a well known result from the standard FEM shows a way to evaluate this norm in the discrete case.

The Galerkin FEM for the Laplace problem:

$$-\Delta u = f \tag{86}$$

leads in the weak form to the solution of following variational form:

$$\int_\Omega \nabla u \cdot \nabla v \, d\Omega = \int_\Omega f v \, d\Omega, \quad \forall v \in H_0^1(\Omega) \tag{87}$$

where $f \in H^{-1}$ and $u, v \in H_0^1$. Introducing a suitable discrete subspace $\mathbf{V}_h \subset H_0^1$, a discrete solution operator $\mathcal{S}_h : H^{-1} \to \mathbf{V}_h \subset H^1$ can be defined. It can be shown that (cf. [20]):

$$(\mathcal{S}_h f, f) = \sup_{\varphi_h \in \mathbf{V}_h \cap H_0^1(\Omega)} \frac{(f, \varphi_h)^2}{|\varphi_h|_1^2} \tag{88}$$

Thus, because the right hand side of Eq. (88) is the restriction of Eq. (12) to $\mathbf{V}_h$:

$$|f|^2_{-1,h} = (\mathcal{S}_h f, f), \quad \forall f \in H^{-1}(\Omega) \tag{89}$$

In the discrete case, this algorithm leads only to a negative seminorm!. To get a norm from the seminorm defined in Eq. (89), it is necessary to add a L_2 term weighted by the characteristic mesh size of the triangulation h:

$$\|f\|^2_{-h} = (\tilde{\mathcal{S}}_h f, f)_0, \qquad \text{with} \qquad \tilde{\mathcal{S}}_h = \alpha h^2 \mathcal{I} + \mathcal{S}_h. \tag{90}$$

Here $\mathcal{I}$ denotes the identity operator and $\alpha > 0$ is a parameter, which can be chosen to change some properties of numerical methods based on this approach. Normally the discrete solution operator $\mathcal{S}_h$ is replaced by a spectrally equivalent operator, which is easier to evaluate.

Because the solution operator $\mathcal{S}^h$ is a full matrix in the discrete case, a direct application would be prohibitively expensive. Therefore most authors propose iterative solvers together with appropriate preconditioners, which can lead to very efficient algorithms (cf. [11]).

For some problems, the reduction of regularity assumptions on the solution is crucial, which will be discussed in more detail in the section about the velocity flux-pressure formulation of the Navier-Stokes equations.

2.4.3 Augmented LSFEM

If the bilinear form of the first order formulation stemming from the partial differential equation is not coercive at all or not coercive in the desired norm, adding seemingly redundant equations can sometimes restore coercitivity.

Probably one of the simplest examples to demonstrate this effect is the LSFEM procedure for the standard diffusion equation. The following explanations are based on the text which can be found in [62]. Considering the diffusion equation:

$$-\nabla \cdot \nabla u = f \quad \text{in} \quad \Omega \tag{91}$$

$$u = g \quad \text{on} \quad \Gamma \tag{92}$$

and assuming for simplicity homogeneous boundary conditions (i.e. $g = 0$), one possible equivalent first order formulation is:

$$\mathbf{p} - \nabla u = 0 \quad \text{in} \quad \Omega \tag{93}$$

$$\nabla \cdot \mathbf{p} = -f \quad \text{in} \quad \Omega \tag{94}$$

$$u = 0 \quad \text{on} \quad \Gamma. \tag{95}$$

Thus the corresponding L_2 functional would be:

$$\mathcal{J}(u,\mathbf{p}) = \|\nabla\cdot\mathbf{p} + f\|_0^2 + \|\nabla u - \mathbf{p}\|_0^2 \tag{96}$$

and is a mapping:

$$\mathcal{J} : H_0^1(\Omega)\times(H(\mathrm{div};\Omega))^n \to \mathbb{R} \tag{97}$$

where n is the space dimension.

Let $\mathcal{B}(\mathbf{u},\mathbf{v})$ denote the associated bilinear form with $\mathbf{u} = (u,\mathbf{p})^T$. Then the following coercitivity estimate holds (the proof can be found in [62]):

$$c_1(\|u\|_1^2 + \|\mathbf{p}\|_{H(\mathrm{div},\Omega)}^2) \le \mathcal{B}(\mathbf{u},\mathbf{u}) \le c_2(\|u\|_1^2 + \|\mathbf{p}\|_{H(\mathrm{div},\Omega)}^2) \tag{98}$$

Using standard finite element functions, this estimate implies suboptimal convergence rates. Extending Eq. (93) by additional equations gives the following system of equations, which is still equivalent to Eq. (91):

$$\begin{aligned}
\mathbf{p} - \nabla u &= 0 \quad \text{in} \quad \Omega && (99)\\
\nabla\cdot\mathbf{p} &= -f \quad \text{in} \quad \Omega && (100)\\
\nabla\times\mathbf{p} &= 0 \quad \text{in} \quad \Omega && (101)\\
u &= 0 \quad \text{on} \quad \Gamma && (102)\\
\mathbf{n}\times\mathbf{p} &= 0 \quad \text{on} \quad \Gamma && (103)
\end{aligned}$$

In two and three dimensions this system of equations is overdetermined in the sense that there are more equations than unknowns. For the mathematical analysis it is hence required to introduce dummy variables. These can be left out for real computations, if they are not part of the boundary conditions which have to imposed to get a well posed problem (cf. [19], [62]). With this formulation it is then possible to prove uniform ellipticity, which implies H^1-coercitivity and optimal convergence rates with standard finite elements.

But the augmented LSFEM is not as advantageous as it might seem. Actually the domain of the first order formulation Eq. (93) is $H_0^1 \times (H(\mathrm{div},\Omega))^n$. The additional equations lead to full H^1-coercitivity but as a consequence the domain of the operator changes to $H_0^1\times(H^1)^n$, which is obviously smaller than the original domain. Hence the augmented formulation finds only the projection of the solution to this smaller space, which might be too small. Therefore the augmented formulations are not of great use as they further increase the regularity demands on the solution (e-Mail correspondence with Prof. Bochev). Similar problems occur also for some formulations of the Stokes and Navier-Stokes equations (cf. section 4).

2.4.4 Streamline Diffusion

In [69] another way is proposed to achieve coercitivity. The equation examined in that article is of convection diffusion type:

$$-\nabla \cdot (\epsilon \nabla u + u\mathbf{c}) + \gamma u = f \quad \text{in}\Omega. \tag{104}$$

where ϵ is a parameter which allows to control the amount of diffusion and $\mathbf{c}$ prescribes a velocity field for the convection. Eq. (104) is transformed into a first order system by introducing the following vector unknown:

$$\mathbf{p} = -\epsilon \nabla u - \mathbf{c}u \tag{105}$$

which then leads to the following equivalent first order system:

$$\mathbf{p} = -\epsilon \nabla u - \mathbf{c}u \tag{106}$$

$$\nabla \cdot \mathbf{p} + \gamma u = f. \tag{107}$$

Now the Eq. (107) is multiplied with $\mathbf{c}$ and the divergence is taken. After that the equation is multiplied with a positive parameter δ. The resulting equation is added to Eq. (107) as the stabilisation term with the stabilisation parameter δ. A similar procedure is applied to Eq. (106).

With these changes it is possible to derive a coercitivity result in a norm containing $\epsilon|u|_1^2 + \delta\|\mathbf{c} \cdot \nabla u\|_0^2$. Numerical results for a simple test case show, that the numerical scheme performs quite well, independent of the parameter ϵ.

2.4.5 FOSLL*

A recent development in the LSFEM is the so called first order system $\mathcal{LL}^*$ (*FOSLL**) method developed in [31]. This method is not really a pure stabilisation but because it is proposed as an alternative to the negative norm methods and furthermore fits well into the series of nonstandardLSFEM methods shown in the previous sections, it will also be presented here.

The basic idea for the FOSLL* method comes from linear algebra. Applying the idea of the LSFEM for the solution of a system of linear equations:

$$\mathbf{A}\mathbf{u} = \mathbf{f}, \tag{108}$$

the normal LSFEM would use matrices of the form $\mathbf{A}^t\mathbf{A}$ to minimise $\|\mathbf{A}\mathbf{u} - \mathbf{f}\|^2$. A dual approach would use matrices of the form $\mathbf{A}\mathbf{A}^t$ instead. Hence the idea is to solve $\mathbf{A}\mathbf{A}^t\mathbf{y} = \mathbf{f}$ and to derive the solution $\mathbf{u} = \mathbf{A}^t\mathbf{y}$ from the solution of the dual problem.

Transferring these ideas to differential operators, the problem:

$$\mathcal{L}u = f \tag{109}$$

is first rewritten as

$$\mathcal{L}\mathcal{L}^* w = f. \tag{110}$$

with the dual variables w. From this the functional for the minimisation process can derived as:

$$\|\mathcal{L}^* w\|^2 - 2(w, f). \tag{111}$$

where the norm depends on the spaces related to the solution of the dual $\mathcal{L}^*$.

This implies that the solution process itself is not different from the one for the normal LSFEM. But the use of the adjoint makes some additional assumptions on the operator $\mathcal{L}$ necessary, which are normally not satisfied by the operators used for the standard LSFEM. Beside the normal requirement that $\mathcal{L}$ and $\mathcal{L}^{-1}$ are continuous, the same must hold for the dual operator $\mathcal{L}^*$. Furthermore u should be in the range of $\mathcal{L}^*$.

Again the convection diffusion equation serves as an test example. In [31] a special form of the operator $\mathcal{L}$ is shown, which satisfies the necessary conditions for the FOSLL*. Astonishingly it is again necessary to introduce some kind of slack variable to make the adjoint a properly determined equation. The necessary equations are derived in 2D and 3D with the necessary coercitivity proofs.

To get the solution u from the dual solution w two ways are proposed. The first one applies the adjoint operator $\mathcal{L}^*$ directly to the dual solution w, while the second casts the resulting system of equations again into the normal LSFEM framework.

An appealing property of the method is, beside the reduced regularity assumptions about the solution u, good multigrid behaviour. This is explained by the resulting operators, where the diagonal contains only second order operators. They dominate the non-diagonal terms and due to their Poisson like character enable good multigrid performance. The article predicts that the convergence rates are independent of the convective part and the coarsest mesh size. Numerical experiments show the predicted convergence behaviour for several simple test cases.

The authors promise to apply their method to equations of more practical importance like the Stokes or Navier-Stokes equations. But the results shown so far already render it an interesting new approach in the area of least squares finite element methods.

3 Fundamental PDEs

For the evaluation of numerical schemes often the same basic partial differential equations are used. Many introductory books about the finite element method use the Laplace operator as an example and the analysis of advection and advection diffusion equations is often an intermediate step before addressing problems of practical importance like the Euler or Navier-Stokes equations. Hence the advection and advection diffusion equations will be analysed in the first part of this chapter. After that the extension of the LSFEM to instationary equations will be discussed.

3.1 Laplace Operator

Probably the equation which exhibits the least numerical problems is the diffusion equation:

$$-\Delta u = f \quad \text{in} \quad \Omega \tag{112}$$

$$\frac{\partial u}{\partial \mathbf{n}} = g \quad \text{on} \quad \Gamma_n \tag{113}$$

$$u = h \quad \text{on} \quad \Gamma_d \tag{114}$$

Despite its simplicity it appears in many applications. The basic phenomenon described by this equation is the stationary state of a diffusion process.

As the standard Galerkin FEM offers optimal approximation properties and leads to a symmetric positive definite matrix with condition number $O(h^2)$ (cf. Chap.6.3.2 in [85]), the LSFEM cannot offer significant advantages. Hence there is not much recent literature about this equation.

One approach, which was already shown in section 2, is to introduce the derivatives of the unknown function u as new unknowns, which leads to the system shown in Eq. (93). For this system only a suboptimal convergence estimate holds with standard finite elements due to the lack of full H^1-coercitivity. Introducing seemingly redundant equations can restore the H^1-coercitivity and thus lead to formally optimal convergence rates (cf. [62], section 2). Both methods would introduce more unknowns and increase the regularity assumptions about the solution ($u \in H^2$ compared to $u \in H^1$ for the Galerkin FEM). These properties are clear disadvantages and hence make the LSFEM not very attractive for this type of equation.

Alternative approaches could use C^1 finite elements and apply the minimisation principle directly (this was done for the closely related advection diffusion equation in [37], cf. also subsection 3.2). Obviously the number of unknowns would not increase in this approach, but the condition number will be $O(h^4)$ and the C^1 finite elements are difficult to implement in higher dimensions. Also the regularity demands on the solution would not change.

3.2 Advection and Advection-Diffusion Equations

Adding a transport term to the Laplace operator leads to the advection-diffusion equation:

$$-\epsilon\Delta u + \mathbf{c}\cdot\nabla u = f \quad \text{in} \quad \Omega \tag{115}$$

$$u = 0 \quad \text{on} \quad \Gamma_D \tag{116}$$

$$\frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \Gamma_N \tag{117}$$

where $\mathbf{c}$ is a prescribed velocity field and ϵ can be used to reduce the amount of diffusion. The relation between diffusion and advection is represented by the Peclet number (cf. [85]). For the Galerkin method the element Peclet number $Pe = h|\mathbf{c}|/2\epsilon$, which includes the characteristic element size h, is of importance (cf. section 8.2 in [85]). If $Pe > 1$, the standard Galerkin method does not work properly, which can be seen by spurious oscillations in the solution (cf. section 8.2 in [85]). From the functional analysis point of view this behaviour can be explained by the loss of coercitivity as the convective part becomes dominant (cf. [24]). Numerically the effect can also be seen in the corresponding discretisation matrix.

To get a working numerical scheme, some stabilisation terms have to be added to the variational form. Numerically these terms add a sufficient amount of diffusion to suppress the spurious oscillations in the solution. The different proposed stabilisations mainly differ in the way the artifical diffusion is added. Accurate schemes like the streamline upwind Petrov Galerkin (SUPG) method and the Galerkin least squares (GLS) method add the diffusion mainly in streamline direction to prevent the solution from becoming too diffusive (cf. [85]). Looking at the mathematical aspects of these methods, they also restore the coercitivity of the variational form. Despite the fact that these methods are very successful, some disadvantages should be noted. The choice of a good stabilisation parameter has a strong impact on the accuracy of the solution and often good choices are hard to find. Another disadvantage is the mathematical convergence order, which is for SUPG and GLS half an order below the optimum (cf. [62]).

The LSFEM formulation for advection diffusion equations does not need these stabilisation techniques as it naturally includes some kind of stabilisation (cf. [62]), which on the other hand might be too much. Therefore the question, if the advection-diffusion equation could be solved accurately and efficiently by the LSFEM is not easy to answer as different authors give different answers to this question.

3.2.1 Standard L_2-LSFEM

In [37] the author compares three numerical methods for solving Eq. (115). It is the classical Galerkin method, the LSFEM, and the least squares stabilised Galerkin method. The LSFEM used in that article is in fact the pure L_2 minimisation which was directly applied to the PDE. Instead of reformulating the equation as a first order system, C^1 finite elements, which guarantee that the second derivative is in L_2, were employed. Besides confirming the well known fact that the Galerkin method leads to spurious oscillations in the solution, the author shows that the LSFEM leads to highly dissipative solutions. An analysis of the resulting element stiffness matrices shows that the LSFEM makes the convective and the diffusive part look like diffusion matrices, which is named as main reason for the high dissipation.

This fact is also explained in the book [48] where the pure convection case ($\epsilon = 0$) is examined. Looking at the LSFEM variational formulation for the 1D pure convection equation:

$$\int_\Omega c^2 \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} = 0 \tag{118}$$

it is clear that it is exactly the same as the variational formulation coming from the Galerkin method for the Laplace operator. Hence mathematically the solution converges to the correct solution, but the convergence rate renders the method useless. Interestingly the LSFEM performs quite well for the instationary case if used together with a normal scheme for time discretisation like the Crank Nicholson method (cf. [48], [49], [62]). The reasons will be explained in subsection 3.3.

This result is also supported by [27], where the LSFEM was used to model convective transport of pollutants in the atmosphere. The method proposed in that article used a normal time discretisation (Trapezoidal Rule, or Crank Nicholson). A well known 2D test case (rotating cone), which simulates an initial distribution inside a rotating convection field was examined. The authors used bilinear and biquadratic ansatz-functions and computed solutions on different meshes. Furthermore they used post-processing filters to steepen smeared gradients. While the bilinear elements did not perform very well, the biquadratic elements gave satisfying results even without the use of the filters. Hence their result supports the thesis that the LSFEM could perform well for convection dominated equations in the instationary case.

3.2.2 Other Least Squares Methods

Besides the approaches based on the L_2 minimisation, several other ideas have been proposed for the solution of the advection diffusion equations.

A negative norm method was used for this type of equation by Bramble et.al. in

[20]. They transform Eq. (115) into the following equivalent first order system:

$$\begin{aligned} \mathbf{p} + \epsilon \nabla u &= 0, \quad \text{in} \quad \Omega && (119)\\ \nabla \cdot \mathbf{p} + \mathbf{c} \cdot \nabla u &= f, \quad \text{in} \quad \Omega && (120)\\ u &= 0, \quad \text{on} \quad \Gamma_D && (121)\\ \mathbf{p} \cdot n &= 0, \quad \text{on} \quad \Gamma_N. && (122) \end{aligned}$$

Eq. (120) is evaluated in the weaker H^{-1} norm, which reduces the regularity demands on the solution. Hence the following functional is minimised:

$$\mathcal{J}(u, \mathbf{p}) = \frac{1}{2} \left(\|\nabla \cdot \mathbf{p} + \mathbf{c} \cdot \nabla u - f\|_{-1}^2 + \|\mathbf{p} + \epsilon \nabla u\|_0^2 \right). \quad (123)$$

For this functional the authors prove the following a priori estimate:

$$c_1(\|\mathbf{p}\|_0^2 + \|u\|_1^2) \leq \|\nabla \cdot \mathbf{p} + \mathbf{c} \cdot \nabla u - f\|_{-1}^2 + \|\mathbf{p} + \epsilon \nabla u\|_0^2 \leq c_2(\|\mathbf{p}\|_0^2 + \|u\|_1^2). \quad (124)$$

This estimate implies coercitivity the the space $H^1 \times L^2$ and adding some assumptions on the approximation spaces, it is possible to show optimal convergence rates. Beside this, estimate Eq. (124) leads to an efficient preconditioner. The variable u can be preconditioned by the Laplace operator and for $\mathbf{U}$ a diagonal preconditioner works well. But as soon as the convection becomes dominant, the performance of the proposed preconditioner degrades according to own numerical tests.

In [53] different L_2 formulations for the convection diffusion equation are analysed. First a formulation coming from the extension of Eq. (99) is analysed:

$$\begin{aligned} \mathbf{p} - \epsilon \nabla u &= 0, \quad \text{in} \quad \Omega && (125)\\ \nabla \times \mathbf{p} &= 0, \quad \text{in} \quad \Omega && (126)\\ \nabla \cdot \mathbf{p} + \mathbf{c} \cdot \nabla u &= -f, \quad \text{in} \quad \Omega && (127)\\ u &= 0, \quad \text{on} \quad \Gamma && (128)\\ \mathbf{n} \times \mathbf{p} &= 0, \quad \text{on} \quad \Gamma && (129) \end{aligned}$$

The convective parts do not change the coercitiveness of the least squares functional, which implies optimal convergence rates of multigrid solvers. Unfortunately, as the article points out, the constants in the coercitivity estimate are not independent of the amount of convection. Thus convergence rates and multigrid performance will degrade as $\mathbf{c}/\epsilon$ grows (assuming standard multigrid components, like normal interpolation operators and Gauss-Seidel smoothing). A more detailed analysis reveals that the system looses its diagonal dominance, which leads to a stronger coupling between u and $\mathbf{p}$ and in the consequence to problems with the multigrid algorithm.

Replacing $\mathbf{c} \cdot \nabla u$ by $\mathbf{c}/\epsilon \cdot \mathbf{p}$ in Eq. (127) removes the coupling between u and $\mathbf{p}$ but increases the coupling between the components of $\mathbf{p}$ which would then again lead to problems with the multigrid algorithm.

The solution proposed in [53] is to introduce an exponential weighting into the system of equations, where functions $\alpha(x, y)$ and $\beta(x, y)$ are chosen such that $\alpha_x = \frac{c_x}{\epsilon}$ and $\beta_y = \frac{c_y}{\epsilon}$. This leads to the following system of equations in 2D (without boundary conditions):

$$e^{\frac{\alpha-\beta}{2}} p_x - e^{-\frac{\alpha+\beta}{2}} \frac{\partial u}{\partial x} = 0 \quad \text{in} \quad \Omega, \tag{130}$$

$$e^{\frac{\beta-\alpha}{2}} p_y - e^{-\frac{\alpha+\beta}{2}} \frac{\partial u}{\partial y} = 0 \quad \text{in} \quad \Omega, \tag{131}$$

$$-e^{\alpha} \frac{\partial p_x}{\partial x} - e^{\beta} \frac{\partial p_y}{\partial y} = \frac{1}{\epsilon} f \quad \text{in} \quad \Omega \tag{132}$$

$$\frac{\partial (e^{\alpha} p_x)}{\partial y} - \frac{\partial (e^{\beta} p_y)}{\partial x} = 0 \quad \text{in} \quad \Omega \tag{133}$$

with $\mathbf{p} = (p_x, p_y)^T$. For the corresponding L_2 functional the coercitivity can be shown in a weighted norm, which is similar to the H^1 norm. Furthermore the numerical experiments show that the multigrid performance is independent of the convection.

The FOSLL* approach which was already mentioned in subsection 2.4.5 and was introduced in [31] also emphasises the importance of the diagonal dominance of the resulting system. With the FOSLL* approach this diagonal dominance can be established in a slightly weaker sense. In that paper the authors show optimal multigrid performance for the convection diffusion equation independent of the amount of diffusion.

In [80] a general discussion of the LSFEM can be found, which also addresses the above mentioned issues. A very detailed discussion of least squares methods for hyperbolic conservation laws can be found in [83]. Especially the behaviour of LSFEM methods in the case of discontinuous solutions and the performance of geometric and algebraic multigrid algorithms are examined in depth. It was found that the discontinuous or nonconforming LSFEM does not offer advantages compared to the LSFEM utilising continuous ansatz spaces. The examination of high order schemes revealed an improved accuracy with still acceptable over- and undershoots in the solution. While the tests with geometric multigrid algorithms did not lead to satisfactory results due to the high anisotropies, the use of a special algebraic multigrid (AMG) algorithm gave satisfactory results.

3.3 Instationary PDEs

Using the standard Galerkin approach for the solution of initial boundary value problems, the time derivative can usually be separated from the spatial differential operator (cf. Eq. (135)). This is not only convenient with respect to using nearly arbitrary time discretisation methods, but also simplifies the analysis significantly as it allows the separate analysis of the spatial and temporal discretisation (cf. [96]). A slightly different situation appears when using the LSFEM method, because the time derivative cannot be easily separated from the other terms in this case. Considering an arbitrary instationary PDE of first order:

$$\frac{\partial u}{\partial t} + \mathcal{L}(u) = f \tag{134}$$

the normal Galerkin method gives:

$$\int_\Omega \frac{\partial u}{\partial t} v \, d\Omega + \int_\Omega \mathcal{L}(u) v \, d\Omega = \int_\Omega f v \, d\Omega \qquad \forall v \in \mathbf{V} \tag{135}$$

For the LSFEM we obtain:

$$\int_\Omega (\frac{\partial u}{\partial t}(\frac{\partial v}{\partial t} + \mathcal{L}(v)) + \mathcal{L}(u)(\frac{\partial v}{\partial t} + \mathcal{L}(v)) \, d\Omega = \int_\Omega f(\frac{\partial v}{\partial t} + \mathcal{L}(v)) \, d\Omega \quad \forall v \in \mathbf{V} \tag{136}$$

where the time derivative is connected with the rest of the partial differential equation. Hence two ways are suitable for solving instationary problems. One approach is to replace the time derivative in the PDE by a finite difference term:

$$\frac{u^{n+1} - u^n}{\Delta t} + (1-\theta)(\mathcal{L}(u^n) - f^n) + \theta(\mathcal{L}(u^{n+1}) - f^{n+1}) = 0. \tag{137}$$

which leads to:

$$u^{n+1} + \Delta t \theta \mathcal{L}(u^{n+1}) = u^n + \Delta t[\theta f^{n+1} + (1-\theta) f^n - (1-\theta)\mathcal{L}(u^n)] \tag{138}$$

After that the least squares principle can be applied straightforward to get a variational statement for the solution at the next timestep. The other way is the use of space-time finite elements, which were used in [81], [8] and [76] to name a few references. Probably also earlier works used space-time finite elements in conjunction with the LSFEM.

3.3.1 Finite Difference LSFEM

Replacing the time derivative by a finite difference is probably the most common approach for solving instationary problems with the LSFEM. It has been used for the

Navier-Stokes equations ([92], [91], [93]), convective transport ([27]), the incompressible Euler equations ([102]) and electromagnetic problems ([101]) to mention just a few applications.

Despite its broad use in applications, few papers are concerned in the mathematical analysis of these instationary formulations. For several equations, the coercitivity estimates should still hold with the additional u in the equation. Hence to prove the well posedness for single timesteps should not be a large problem. Nevertheless further examinations, which analyse the complete time evolution of the solution, would probably be more difficult and have to our knowledge not been done yet.

Another issue is the used time integration scheme. In the semi-discrete case (here in the sense that the equation is continuous in space and discrete in time) the time integration scheme is the θ-method, which includes the popular back- and forward Euler methods and the trapezoidal rule which is equivalent to the Crank-Nicholson scheme. But after applying the least squares principle to Eq. (137) the resulting time integration scheme has changed! This can be seen on the instationary convection equation:

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0. \tag{139}$$

Using Green's theorem, the variational formulation coming from the LSFEM can be transformed into the following equivalent variational statement (cf. [62]):

$$((1 - \theta\Delta t\, c\frac{\partial}{\partial x})[(1 + \theta\Delta t\, c\frac{\partial}{\partial x})(u^{n+1} - u^n) + \Delta t\, c\frac{\partial u^n}{\partial x}], v). \tag{140}$$

This variational form can be interpreted as a Galerkin approach for the following partial differential equation:

$$\frac{u^{n+1} - u^n}{\Delta t} + c\frac{\partial u^n}{\partial x} - \theta\Delta t c^2\frac{\partial^2 u^n}{\partial x^2} - \theta^2\Delta t^2\frac{\partial^2}{\partial x^2}(\frac{u^{n+1} - u^n}{\Delta t}) = 0. \tag{141}$$

Setting $\theta = 1/2$ leads to a formulation, which is similar to the Taylor-Galerkin approach for the convection equation (cf. [62]).

A similar analysis was performed in [48] and [49]. In [48] an argument similar to Eq. (141) is used as an explanation why the LSFEM, which gives poor results for the stationary convection equation, performs quite well in the instationary case. The analysis, which was cited above was originally published in [41].

The available numerical results for instationary problems indicate that the stability properties of the θ-methods seem to be conserved if it is used in conjunction with the LSFEM. But it is not clear, if this property holds for all types of equations.

3.3.2 Space-time LSFEM

A comparison of different algorithms for time discretisation, which can be found in [110] and is based on works from the seventies [109], [100], shows that the least squares approximation of transient problems yields a good accuracy. In those articles the L_2 LSFEM principle was applied to ordinary differential equations.

For convection problems a space-time LSFEM was introduced by Nguyen and Reynen in 1984 ([81]). They also showed that this approach does not need any special treatment like upwinding or the Taylor-Galerkin method. But this approach was found to be less accurate and more dissipative than an approach based on using the θ-method for time integration (cf. [49]).

In [8] the space-time LSFEM was used for the instationary Navier-Stokes equations in the velocity-stress-pressure formulation and in [66] a similar scheme was proposed for the velocity-vorticity-pressure formulation (cf. section 4 and section 6). Unfortunately, in [8] the time stepping was only used to achieve the stationary state of a driven cavity example. Therefore the paper makes no statements about the time accuracy of the LSFEM.

A recent publication from Majidi and Starke ([76] and [77]) is about the space time LSFEM for parabolic problems. They want to use the space-time LSFEM to utilise the ”built-inërror indicator of the LSFEM. For the mathematical analysis they consider a numerical procedure which splits the space-time domain into time slabs. One of these slabs, consisting of only one element in time direction, is then used for the analysis. Linear ansatz-functions are used for the time dimension and are inserted into the integral equations. The integral in time direction is solved by Simpson’s rule, which is exact for polynomials of order up to two. Therefore this approach is exact, if the right hand side of the PDE is assumed to be constant in time.

In the first article [76] the stability of the resulting numerical method is analysed. The authors show coercitivity of the bilinear form for one timestep and also stability for the complete instationary problem. Numerical tests and details about the adaptive algorithm are published in the second part [77]. The numerical results confirm the theoretical results.

Looking again at the variational principle coming from the space-time LSFEM for

$$\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0 \quad \text{in} \quad \Omega \tag{142}$$

the following integral statement can be derived:

$$\int_{\Omega\times[0,T]} c\left(\frac{\partial u}{\partial t}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial v}{\partial t}\right) + \frac{\partial u}{\partial t}\frac{\partial v}{\partial t} + c^2\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}\,dx\,dt \tag{143}$$

It is easy to see that the last two terms in Eq. (143) are similar to the terms obtained when applying the normal Galerkin method to an anisotropic diffusion equation. These

terms can be interpreted as stabilisation in the streamline direction. But according to the results obtained in [49] the formulation is probably över-stabilised". Normally in numerics more stability is coupled with less accuracy (cf. [57]), which explains the low accuracy of the space-time LSFEM. Another disadvantage is the increased number of unknowns and a condition number, which becomes larger with reduced timestep size. This is a completely different behaviour compared to the standard time integration schemes, where a smaller timestep normally leads to a reduced condition number!

Own numerical tests with the space-time LSFEM for the Navier-Stokes equation revealed even more negative properties of the space-time LSFEM. For a simple test case, the mass conservation became worse with smaller timesteps, which also indicates that something is probably wrong with the numerical scheme.

4 The Stokes Equations

The incompressible Stokes equations are:

$$-\nu\Delta\mathbf{u} + \nabla p = f \quad \text{in} \quad \Omega \tag{144}$$

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in} \quad \Omega. \tag{145}$$

These equations must be accompanied by some boundary conditions, which are often hard to find, if they should correspond to a physical setting (one example is the outflow boundary condition, which is still subject of discussion, cf. [57]). In the mathematical analysis the following condition is often added:

$$\int_\Omega p \, d\Omega = 0. \tag{146}$$

It ensures a zero mean pressure and circumvents problems with the pressure unknown, which is only determined up to a constant by Eq. (144).

The first section will summarise some properties and applications of the Stokes equations and give a short introduction to the problems which appear if the standard Galerkin method is used for this problem. Different first order LSFEM formulations will then be presented in the subsequent sections together with their mathematical properties.

4.1 Theory and Applications

Probably most often the Stokes equations appear as a preliminary step in the numerical analysis before the full nonlinear Navier-Stokes equations are considered. The missing nonlinear convective terms make the analysis easier. Thus they are equivalent to the Navier-Stokes equations with $Re = 0$ and hence describe fluid flow in a very viscous medium.

Another application comes from the equations of linear elasticity:

$$-\mu\Delta\mathbf{u} - (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) = f \quad \text{in} \quad \Omega \tag{147}$$

where λ and μ are the Lamé constants. With $p = \nabla \cdot \mathbf{u}$ we obtain the following system of equations:

$$-\mu\Delta\mathbf{u} + (\lambda + \mu)\nabla p = f \quad \text{in} \quad \Omega \tag{148}$$

$$\nabla \cdot \mathbf{u} = p \quad \text{in} \quad \Omega \tag{149}$$

which is a special form of the compressible Stokes equations. The general form can be written as:

$$-\nu\Delta\mathbf{u} + \nabla p = \tilde{f} \quad \text{in} \quad \Omega \tag{150}$$

$$\nabla \cdot \mathbf{u} = p \quad \text{in} \quad \Omega \tag{151}$$

with $\nu = \frac{\mu}{\lambda+\mu}$ and $\tilde{f} = \frac{f}{\lambda+\mu}$. In this formulation the unknown function $\mathbf{u}$ describes the structural displacements.

The Galerkin variational principle for Eq. (144) is:

$$a(\mathbf{u}, \mathbf{v}) + b(p, \mathbf{v}) = 0 \qquad \forall v \in \mathbf{V} \tag{152}$$

$$b(q, \mathbf{u}) = 0 \qquad \forall q \in \mathbf{Q} \tag{153}$$

with the following bilinear forms:

$$a(\mathbf{u}, \mathbf{v}) = \nu \int_\Omega \nabla \mathbf{u} \cdot \nabla \mathbf{v} \, d\Omega \tag{154}$$

$$b(q, \mathbf{v}) = \int_\Omega q \nabla \cdot \mathbf{v} \, d\Omega \tag{155}$$

In contrast to the variational principle for the Laplace operator, which corresponds to a minimisation problem, the variational principle for the Stokes equations can be seen as a saddle point problem. Due to the lack of a minimisation principle, the Galerkin variational principle does not work properly if used with equal order ansatz-functions for the velocity and pressure ansatz-functions (cf. [57], [110], [85], [55]).

Galerkin methods also depend on estimates, which guarantee coercitivity of the bilinear forms. For the Stokes equations these estimates are a bit more complicated. Suitable function spaces for $\mathbf{u}$ and p are $\mathbf{V} = H^1(\Omega)$ and $\mathbf{P} = L_2(\Omega)$ with a zero mean condition (cf. [55]). For the bilinear form $a(\mathbf{u}, \mathbf{v})$ it is possible to show the necessary coercitivity estimate. The bilinear form $b(q, \mathbf{v})$ is not coercive according to the standard definition. But the following estimate can be used to show the well posedness of the Galerkin formulation:

$$\inf_{q \in L_0^2(\Omega)} \sup_{\mathbf{v} \in H_0^1(\Omega)^n} \frac{b(q, \mathbf{v})}{\|q\|_0 \|\mathbf{v}\|_1} \geq \beta \quad \text{for} \quad \beta > 0. \tag{156}$$

It is known as the *inf-sup* or *Ladyzhenskaya-Babuška-Brezzi* (LBB) condition (cf. [55]). A consequence for numerical methods based on this variational principle is the need for special discrete subspaces, which satisfy the discrete counterpart of Eq. (156). Therefore in contrast to the Galerkin formulation of the Laplace operator, where it was sufficient that $\mathbf{V}_h \subset H_0^1$, the inclusions $\mathbf{V}_h \subset H_0^1$ and $\mathbf{P}_h \subset L_0^2$ alone are not sufficient to guarantee stability and accuracy. Discrete pairs of subspaces, which satisfy Eq. (156) are not easy to find and often inconvenient for the implementation. A selection of possible element pairs can be found in most standard textbooks like [110] or [57].

Alternatively the variational principle can be enriched by so called *stabilisation* terms, which lead to a coercive bilinear form. A summary of these methods can be found in [85].

The least squares finite element method circumvents the problems with the inf-sup condition because the original saddle point problem is cast into a minimisation problem. But the bilinear form stemming from the least squares approach also has to be coercive, which can lead to other problems.

It was pointed out in the introduction that first order methods are currently the favoured research direction because they seem to be the most practical implementation of a least squares principle. Clearly Eq. (144) has some second order terms, which make the conversion into a first order system inevitable.

One exception from this rule is a method proposed by Bramble et.al. who apply a negative norm directly to the residuum of the second order equation ([21]). Due to their negative norm approach, no special elements are required because the second derivative of standard finite elements is still in H^{-1} and can therefore be used. The method performs very well even in the case of an incompressible structure but the implementation is more complicated as the jump terms across the element edges have to be included into the minimised functional.

Currently several first order approaches have been used and/or analysed in literature:

- Velocity-Vorticity-Pressure
- Velocity-Stress-Pressure
- Velocity-Velocity Flux-Pressure

Although these first order systems are equivalent in the continuous case, their mathematical properties differ. This will be shown in the next sections, where the main results for these equations will be summarised.

4.2 Velocity-Vorticity-Pressure Formulation

Introducing the vorticity $\omega = \nabla \times \mathbf{u}$ as an additional variable leads to the velocity-vorticity-pressure formulation (abbreviated as $u\omega p$-formulation):

$$\nu \nabla \times \omega + \nabla p = \tilde{\mathbf{f}} \quad \text{in} \quad \Omega \tag{157}$$

$$\nabla \cdot \omega = 0 \quad \text{in} \quad \Omega \tag{158}$$

$$\omega - \nabla \times \mathbf{u} = 0 \quad \text{in} \quad \Omega \tag{159}$$

$$\nabla \cdot \mathbf{u} = p \quad \text{in} \quad \Omega \tag{160}$$

In 2D the vorticity vector reduces to a single scalar, while in 3D the vorticity introduces three new unknowns. Thus Eq. (158) will only appear in the 3D case.

It is probably the most popular least squares formulation for the Stokes and Navier-Stokes equations. A large number of articles examine the mathematical (cf. [64], [33],

[12], [14], [15], [16], [19] and more) and practical (cf. [92], [91], [93], [62], [65], [102] and more) properties of this formulation.

For the 2D case the ADN Theory can be applied directly to the equations as we have an even number of equations and unknowns. In 3D it is necessary to introduce an additional equation and a slack variable φ, which could be left out in computations with the right boundary conditions (cf. [19], [62]). Then the first order system reads:

$$\begin{aligned} \nu\nabla\times\omega+\nabla p &= \tilde{\mathbf{f}} \quad \text{in} \quad \Omega && (161)\\ \nabla\cdot\omega &= 0 \quad \text{in} \quad \Omega && (162)\\ \nabla\times\mathbf{u}+\nabla\varphi-\omega &= 0 \quad \text{in} \quad \Omega && (163)\\ \nabla\cdot\mathbf{u} &= 0 \quad \text{in} \quad \Omega && (164) \end{aligned}$$

The analysis was first done in [64]. Later Bochev and Gunzburger found out that the analysis was not completely correct (cf. [19]).

The ADN-Theory requires that the *principal part* of the equations which must be uniformly elliptic is accompanied by boundary conditions which satisfy the so called *complementing condition* (cf. [2]). Depending on the principal part which is chosen for the analysis the following two a priori estimates can be established ([19]):

$$\begin{aligned} &\|\omega\|_{q+1}+\|p\|_{q+1}+\|\mathbf{u}\|_{q+1}\leq \\ &\qquad c_1(\|\nu\nabla\times\omega+\nabla p\|_q+\|\nabla\times\mathbf{u}-\omega\|_q+\|\nabla\cdot\mathbf{u}\|_q) \end{aligned} \tag{165}$$

$$\begin{aligned} &\|\omega\|_{q+1}+\|p\|_{q+1}+\|\mathbf{u}\|_{q+2}\leq \\ &\qquad c_2(\|\nu\nabla\times\omega+\nabla p\|_q+\|\nabla\times\mathbf{u}-\omega\|_{q+1}+\|\nabla\cdot\mathbf{u}\|_{q+1}) \end{aligned} \tag{166}$$

But each of these estimates needs the right boundary conditions to hold. A simple counterexample (cf. [19]) reveals that Eq. (165) does not hold with a a pure velocity boundary condition. Prescribing the pressure and the normal velocity would satisfy the complementing condition and thus lead to an optimally accurate method which requires only pure L_2 minimisation.

This is the reason that Jiang derives a set of six boundary conditions in [62], which would satisfy the complementing condition. He also shows with a numerical example that using pure velocity boundary conditions results in suboptimal convergence rates in contrast to his proposed boundary conditions. Although this is true, the proposed boundary conditions may render the method useless for practical applications. Using the Stokes equations for linear elasticity, one would normally like to prescribe only the displacements on the boundary and not variables like the vorticity.

An alternative is the use of estimate Eq. (166) where the velocity boundary conditions satisfy the complementing condition. Unfortunately this estimate requires for $q=0$ the minimisation of H^1 norms, which can be done only by using C^1 finite

element spaces. These spaces have two disadvantages. First their implementation in 2 dimensions is already very demanding and nearly impossible in 3 or more dimensions. The other disadvantage is related to the numerical solution of the resulting system of linear equations. In conjunction with the LSFEM the C^1 spaces lead to a condition number of $O(h^{-4})$ and will therefore lead to poor performance of iterative solvers unless an efficient preconditioner is found.

One way to circumvent this problem is the use of weighted norms. It is possible to show that all discrete norms are equivalent (cf. [42]). The equivalence factor between the H^1 and the L_2 norm is h^{-1} on a discrete finite element subspace with h being a parameter characterising the element size. So the following functional has to be minimised:

$$\mathcal{J}(\omega, \mathbf{u}, p) = \frac{1}{2}\left(\|\nu\nabla\times\omega + \nabla p - \tilde{\mathbf{f}}\|_0^2 + h^{-2}\|\nabla\times\mathbf{u} - \omega\|_0^2 + h^{-2}\|\nabla\cdot\mathbf{u}\|_0^2\right) \tag{167}$$

Because this functional uses different spaces than the ones appearing in Eq. (166), the functional is not coercive in the usual sense. But it is possible to establish the following stability estimate which shows the convergence of the discretisation based on Eq. (167) (cf. [19]):

$$\|\omega-\omega_h\|_{q+1} + \|p-p_h\|_{q+1} + \|\mathbf{u}-\mathbf{u}_h\|_{q+2} \leq c_1 h^{\tilde{d}-q}(\|\omega\|_{\tilde{d}+1} + \|p\|_{\tilde{d}+1} + \|\mathbf{u}\|_{\tilde{d}+2}) \tag{168}$$

Here $-d \leq q \leq -1$ and $\tilde{d} = \min\{d, l\}$, where the index l is related to the function spaces (for details cf. [19]). To achieve optimality, the polynomial degree of the ansatz-functions for the discretisation of $\mathbf{u}$ must be taken one order higher than for p and ω.

As already mentioned in subsection 2.4.2 the general character of estimate Eq. (166) allows to choose a functional with negative norms for minimisation. This approach was analysed in [33] with application to linear elasticity. That paper does not utilise ADN theory but directly shows ellipticity and continuity of the functional in a product norm.

One disadvantage appearing generally in all finite element methods for the incompressible Stokes and Navier-Stokes equations is the lack of local mass conservation. In the LSFEM even global mass conservation is violated, because the equation which ensures that the velocity field is divergence free is just another part in the minimised functional. Hence if the other equations dominate the residuum the mass conservation can become quite weak. This is shown in [39] where a Stokes flow around a cylinder is simulated with the LSFEM. Although the author uses the weighted LSFEM functional Eq. (167), the mass conservation is violated at the parts where the channel is narrowed by the cylinder, which can seen in the velocity fields, which are presented in that article. It comes out that more mass flows into the domain than goes by the cylinder. Thus something is lost.

Chang shows a way how to circumvent this disadvantage (cf. [39]). He adds an additional constraint to the system of equations which ensures mass conservation in every triangle. The constraints are then implemented using Lagrange multipliers. This allows the interpretation of his method as a kind of penalty formulation which unfortunately leads also to a saddle point problem. So the disadvantage of that method is that one loses some of the nice properties originally introduced by the LSFEM.

Furthermore Deang and Gunzburger repeated similar numerical test in [47] and were not able to confirm the findings of Chang. In their article several different combinations of boundary conditions and functionals are examined. To emphasise the mass conservation an additional weight, is introduced into the equation which ensures mass conservation:

$$\mathcal{J}(\omega,\mathbf{u},p)=\frac{1}{2}(\|\nu\nabla\times\omega+\mathrm{grad}\,p-\tilde{\mathbf{f}}\|_0^2+h^{-2}\|\nabla\times\mathbf{u}-\omega\|_0^2+Kh^{-2}\|\nabla\cdot\mathbf{u}\|_0^2) \quad (169)$$

Already small values for K (i.e. $K > 5$) improved the mass conservation. Higher values of K did not lead to further improvements. Additional tests examined the effect of different choices for the mesh dependent parameter h in Eq. (167). Tests were performed with element-wise weighting, which corresponded to the diameter of the elements and with an average, which was computed over all elements of the computational grid.

4.3 Velocity-Stress-Pressure Formulation

Introducing the stresses as new unknowns gives another popular first order formulation:

$$\begin{aligned}
\sqrt{2\nu}\,\nabla\cdot\mathbf{T}-\nabla p &= \mathbf{f} \quad \text{in} \quad \Omega && (170)\\
\nabla\cdot\mathbf{u} &= 0 \quad \text{in} \quad \Omega && (171)\\
\mathbf{T}-\sqrt{2\nu}\,\mathbf{E}(\mathbf{u}) &= 0 \quad \text{in} \quad \Omega. && (172)
\end{aligned}$$

Here $\mathbf{T} = \sqrt{2\nu}\mathbf{E}(\mathbf{u})$ denotes the stress tensor scaled by $\sqrt{2\nu}$ with $\mathbf{E}(\mathbf{u}) = (1/2)(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$ being the linearised strain tensor. In 2D the system has 6 equations and 6 unknowns and in 3D 10 equations and the same number of unknowns. Jiang points out that these variables are not independent due to the incompressibility condition (cf. [62]). His conclusion is that the stress formulation actually has only 9 unknowns and equations. Nevertheless, the mathematical analysis of the formulation was done using the ADN theory in [16] and the main results can also be found in [17].

While the right boundary conditions can lead to H^1-coercitivity in the $u\omega p$-formulation, Eqs. (170–172) cannot be made elliptic in the sense of Petrovsky (cf. p.42 [51]) and hence fail to be H^1-coercive (cf.[16]). But it is possible to define a functional which is norm equivalent in the discrete case, similar to the one used for the

formulation using the vorticity:

$$\mathcal{J}(\mathbf{T},\mathbf{u},p)=\frac{1}{2}(h^{-2}\|\mathbf{T}-\sqrt{2\nu}\mathbf{E}(\mathbf{u})\|_0^2+h^{-2}\|\nabla\cdot\mathbf{u}\|_0^2+\|\sqrt{2\nu}\nabla\cdot\mathbf{T}-\nabla p-\mathbf{f}\|_0^2) \quad (173)$$

With this functional it is then again possible to establish optimal convergence results using weighted least squares functionals. Also the use of negative norm methods seems to be possible but was not examined yet.

One disadvantage of the displacement-stress-pressure formulation is the high number of equations and unknowns (6 Unknowns in 2D and 10 Unknowns in 3D), which induces higher computational costs without bringing significant advantages, if the stresses are not of interest.

According to the available literature, this formulation is rarely used for the Stokes equations. Most articles examine the extension of this formulation to the Navier-Stokes equations (cf. section 6).

4.4 Velocity-Velocity Flux-Pressure Formulation

Another first order formulation was proposed by Chang ([38]), who called the method acceleration-pressure formulation. The full displacement gradient tensor is introduced as a new unknown:

$$-\nu(\nabla^T\mathbf{U})^T+\nabla p = \mathbf{f} \quad \text{in} \quad \Omega \quad (174)$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in} \quad \Omega \quad (175)$$

$$\mathbf{U}-\nabla\mathbf{u} = 0 \quad \text{in} \quad \Omega \quad (176)$$

Obviously this makes the resulting system of equations significantly larger than the previously shown first order systems. In the 2D case the resulting system has 7 unknowns and in the 3D case already 13 unknowns. Nevertheless this first order system has some properties which render it an alternative to the other first order systems.

If the system Eqs. (174–176) is accompanied by a velocity boundary condition, the resulting functional is not coercive in appropriate norms and hence the method minimising only the following functional:

$$\mathcal{J}(\mathbf{u},\mathbf{U},p)=\frac{1}{2}(\|-\nu(\nabla^T\mathbf{U})^T+\nabla p-\mathbf{f}\|_0^2+\|\nabla\cdot\mathbf{u}\|_0^2+\|\mathbf{U}-\nabla\mathbf{u}\|_0^2) \quad (177)$$

is suboptimal (cf. [11]).

Two ways exists to establish again the desired coercitivity results. First one might choose again different norms for the different parts of the equation, which results then in the following functional (cf. [11]):

$$\mathcal{J}(\mathbf{u},\mathbf{U},p)=\frac{1}{2}(\|-\nu(\nabla^T\mathbf{U})^T+\nabla p-\mathbf{f}\|_{-1}^2+\|\nabla\cdot\mathbf{u}\|_0^2+\|\mathbf{U}-\nabla\mathbf{u}\|_0^2) \quad (178)$$

Again either weighted norms or discrete H^{-1} method can be applied to this functional. To achieve optimal accuracy the ansatz-functions for u must also be one order higher than those for $\mathbf{U}$ and p.

The other way is to add some seemingly redundant equations, to get a system which is fully H^1-coercive. For the velocity-velocity flux-pressure formulation these equations are (cf. [34]):

$$\nabla(tr\mathbf{U}) = 0 \quad \text{in} \quad \Omega \tag{179}$$

$$\nabla \times \mathbf{U} = 0 \quad \text{in} \quad \Omega \tag{180}$$

$$\mathbf{n} \times \mathbf{U} = 0 \quad \text{on} \quad \Gamma. \tag{181}$$

(The first one describes the gradient of the divergence, which is obviously zero due to the fact that the field should be divergence free. In the second equation the symmetry of second derivatives is expressed in a compact manner ($u_{xy} = u_{yx}$)). The resulting functional is then:

$$\mathcal{J}(\mathbf{u},\mathbf{U},p) = \frac{1}{2}(\|-\nu(\nabla^T\mathbf{U})^T+\nabla p-\mathbf{f}\|_0^2+\|\nabla\cdot\mathbf{u}\|_0^2+\|\mathbf{U}-\nabla\mathbf{u}\|_0^2+\|\nabla(tr\mathbf{U})\|_0^2+\|\nabla\times\mathbf{U}\|_0^2) \tag{182}$$

Unfortunately, these additional equations lead to more equations than unknowns and therefore the system is not of ADN type anymore. But it is possible to show full H^1-coercitivity for this pure L_2 functional. Hence the method achieves optimal accuracy for equal order interpolation of all unknowns (cf. [12], [11], [34]).

One drawback of this method is that the additional equations restrict the function spaces which are used to find a solution (cf. subsection 2.4.3). Hence instead of the original problem related spaces, the space $H^1 \times H^1 \times H^1$ is used to find a solution, which then represents only a projection of the real solution onto this space. Numerical experiments performed with the counterpart of this formulation for the nonlinear Navier-Stokes equations revealed that it has problems to converge to the correct solution in case of discontinuous boundary conditions (cf. [11]).

4.5 Summary

As the original saddle point problem is cast into an equivalent minimisation problem, the LSFEM circumvents the LBB condition. But as the mathematical analysis revealed, the bilinear forms of nearly all first order formulations are not fully H^1-coercive and hence require either mesh dependent weights or the use of negative norms.

Therefore none of the three shown first order formulations has a significant advantage over the other formulations. All require weights and different polynomial degrees for the unknowns to achieve optimal theoretical convergence rates. Thus it seems to

be best to use the formulation with the least number of unknowns, which is clearly the velocity-vorticity-pressure formulation.

The only exception is the augmented velocity-velocity flux-pressure formulation, where full H^1-coercivity was shown. Therefore this formulation should give optimal Multigrid performance and allows the use of elements with the same polynomial degree for all unknowns. But the results presented in [11] show that the high regularity demands on the solution are not only of theoretical interest as they can lead to a significant underestimation of the real solution.

5 Equations of Linear Elasticity

A Galerkin variational principle works very well for the solution of the equations of linear elasticity as long as the material does not approach the incompressible limit. This phenomenon is called *locking*. It is connected to the standard finite element basis functions, which have a special property. As the condition $\|\nabla \cdot \mathbf{u}_h\|_0 = 0$ is approached also $\|\mathbf{u}\|_1 \to 0$ (cf. [24]). Hence the only admissible solution is 0 in the incompressible limit. This property makes the structure far too stiff. As linear elasticity is very important for engineering applications, several methods were developed to circumvent these limitations. The possibilities include special basis functions, stabilisation techniques, and mixed finite element methods (cf. [9]).

The literature about first order LSFEM for the equations of linear elasticity can currently be divided into three main directions. As it was already mentioned in section 4, the equations of linear elasticity are equivalent to the compressible Stokes equations. Thus all methods for the Stokes equations are suitable for the equations of linear elasticity, at least theoretically. These methods will be listed in the first section. After that a comparatively new approach will be shown, which uses the displacement gradient as unknown. The last section will then cover a method which uses the real stress tensor as unknown.

5.1 Formulations Based on the Stokes Equations

One problem with the $u\omega p$-formulation for the Stokes equations is the need for mesh dependent weights. While this is tolerable for CFD-applications, where the material parameters are the same in the whole domain in most cases, it is a real disadvantage in structural mechanics, where also different material parameters could appear in different regions of the domain. A functional, which avoids the mesh dependent weights by the use of a discrete negative norm was examined in [33]:

$$\mathcal{J}(\mathbf{u},\omega,p) = \|f - (\nu\nabla\times\omega + (1+\nu\delta)\nabla p\|_{-1}^2 + \nu^2\|\nabla\times\mathbf{u} - \omega\|_0^2 + \nu^2\|\nabla\cdot\mathbf{u} + \delta p\|_0^2. \tag{183}$$

Here $\nu = \frac{\mu}{\lambda+\mu}$ and δ is used to switch between the incompressible Stokes equations $\delta = 0$ and the equations of linear elasticity $\delta = 1$. Functional Eq. (183) can be efficiently preconditioned by standard elliptic preconditioners and hence leads to an efficient numerical method (cf. [33]).

Two other Stokes formulations, which lead to optimal accuracy without mesh dependent weights are the augmented uUp-formulation (Eq. (182)) and the formulation with a negative norm (Eq. (178)). Their application to linear elasticity is examined in [34]. Due to the coercitivity results, which are independent of the Lamé constants, efficient preconditioners are available for this formulation as well.

Another least squares formulation, which is based on a Stokes like formulation for the equations of linear elasticity is the negative norm approach shown in [21]. Here the minimised functional includes the second derivatives, which are still in H^{-1} if normal C_0 finite elements are considered. Without going into detail, the essential functional is:

$$\mathcal{J}(\mathbf{u},p) = \|\mathcal{L}(\mathbf{u},p)\|_{-1} + \mu_0\|\nabla\cdot\mathbf{u} + \gamma p\|_0 + \|\sigma_n\|_{h,\Gamma_N} + \|[\sigma_n]\|_{h,I} + \|\mathcal{K}(\mathbf{u},p)\|_h \tag{184}$$

where $\|\cdot\|_{h,\Gamma_N}$ and $\|\cdot\|_{h,I}$ denote special norms, which are defined on the element edges of the triangulation. $[\sigma_n]$ denotes the jump term of the stresses across the element boundary, $\mathcal{L}$ is the differential operator associated with the divergence of the stress tensor and $\mathcal{K}$ belongs to $\gamma p + \nabla\cdot\mathbf{u} = 0$, which is measured in a L_2 norm, weighted by the characteristic element size h. The parameter μ_0 provides a bound for the Lamé parameter μ, hence $0 < \mu_0 \leq \mu(x) \leq c_1\mu_0$ for an arbitrary $c_1 > 0$

For this functional Bramble et.al. [21] prove the following estimate:

$$c_2(\mu_0\|\mathbf{u}\|_1 + \|p\|_0) \leq \mathcal{J}(\mathbf{u},p) \tag{185}$$

for some $c_2 > 0$ independent of h. A similar upper bound is easy to find and leads to an efficient elliptic preconditioner which can be used to construct a numerical scheme with an overall complexity of $O(n)$, where n is the number of unknowns. Numerical tests confirm the results (cf. [21]). An interesting feature of this method is, that in contrast to the other least squares methods, discontinuous piecewise constant functions could be used as approximation space for the pressure.

5.2 Displacement, Displacement Gradient

It is not clear, whether the first order formulation, which introduces the displacement gradient $\mathbf{U} = \nabla\mathbf{u}$ as a new unknown, was first proposed by Yang et.al. (cf. [105]) or by Cai et.al. (cf. [32] and [67]) as those articles do not refer to each other and appeared approximately at the same time. The equations of linear elasticity then become together with mixed boundary conditions (cf. [30]):

$$\mathbf{U} - \nabla\mathbf{u} = 0 \quad \text{in} \quad \Omega \tag{186}$$

$$-\nabla\cdot\mathbf{A}\mathbf{U} = \mathbf{f} \quad \text{in} \quad \Omega \tag{187}$$

$$\nabla\times\mathbf{U} = 0 \quad \text{in} \quad \Omega \tag{188}$$

$$\mathbf{n}\cdot\mathbf{A}\mathbf{U} = 0 \quad \text{on} \quad \Gamma_n \tag{189}$$

$$\mathbf{t}\cdot\mathbf{U} = 0 \quad \text{on} \quad \Gamma_d \tag{190}$$

Here $\mathbf{t}$ denotes the counterclockwise oriented unit tangent vector on the boundary. The matrix $\mathbf{A}$ represents the material parameters and is in 2D:

$$\mathbf{A} = \lambda \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} + 2\mu \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{191}$$

where λ and μ are the Lamé parameters. This basic formulation leads to different numerical schemes. Because $\mathbf{u}$ appears only in Eq. (186) it is possible to remove it from the system of equations and recover it later by solving only Eq. (186). This *two-stage* approach can be implemented with two different functionals for Eqs. (187–188). The first one uses negative norms and was proposed in [32]:

$$\mathcal{J}_{-1}(\mathbf{U}) = \|\mathbf{f} + \nabla \cdot \mathbf{A}\mathbf{U}\|^2_{-1} + \|\nabla \times \mathbf{U}\|^2_{-1}. \tag{192}$$

Alternatively the following functional which utilises only L_2 norms can be used (cf.[105] and [32]):

$$\mathcal{J}_0(\mathbf{U}) = \|\mathbf{f} + \nabla \cdot \mathbf{A}\mathbf{U}\|^2_0 + \|\nabla \times \mathbf{U}\|^2_0. \tag{193}$$

The last functional, which was proposed in [32] solves for $\mathbf{u}$ and $\mathbf{U}$ simultaneously:

$$\mathcal{J}(\mathbf{U}, \mathbf{u}) = \|\mathbf{f} + \nabla \cdot \mathbf{A}\mathbf{U}\|^2_0 + \|\nabla \times \mathbf{U}\|^2_0 + \langle \mathbf{A}(\mathbf{U} - \nabla\mathbf{u}), \mathbf{U} - \nabla\mathbf{u}\rangle_0 \tag{194}$$

These three functionals were extended to the 3D case in [67].

In [105] using the theory developed by Wendland for the LSFEM (cf. [98]), the following estimates are presented for the 2D case:

$$\|\mathbf{U}\|_{r+1} \leq c_1(\|\mathcal{L}\mathbf{U}\|_r + \|\mathcal{R}\mathbf{U}\|_{r+1/2}) \tag{195}$$

where $r \geq 0$ and $\mathcal{L}$ denotes Eqs. (187–188) and $\mathcal{R}$ denotes the boundary terms Eqs. (189–190). Clearly $\mathbf{U}$ must have sufficient regularity for this estimate to hold, i.e. $\mathbf{U} \in H^{r+1}$. In the other parts this estimate is used to establish optimal convergence rates for the stresses with standard finite elements. These results are confirmed by numerical examples.

Slightly different estimates are derived in [32]. Here the boundary conditions do not appear in the estimates:

$$c_2(\|\mathbf{U}\|^2_1 + \lambda^2\|\nabla tr\mathbf{U}\|^2_0) \leq \mathcal{J}_0(\mathbf{U}) \leq c_3(\|\mathbf{U}\|^2_1 + \lambda^2\|\nabla tr\mathbf{U}\|^2_0). \tag{196}$$

For functional $\mathcal{J}_{-1}$ we have:

$$c_4(\|\mathbf{U}\|^2_0 + \lambda^2\|tr\mathbf{U}\|^2_0) \leq \mathcal{J}_{-1}(\mathbf{U}) \leq c_5(\|\mathbf{U}\|^2_0 + \lambda^2\|tr\mathbf{U}\|^2_0). \tag{197}$$

Finally for $\mathcal{J}$:

$$c_6(\|\mathbf{U}\|_1^2 + \lambda^2\|\nabla tr\mathbf{U}\|_0^2 + \lambda\|tr\mathbf{U}\|_0^2 + \|\mathbf{A}^{1/2}\nabla\mathbf{u}\|_0^2) \leq \mathcal{J}(\mathbf{U},\mathbf{u}) \leq c_7(\|\mathbf{U}\|_1^2 + \lambda^2\|\nabla tr\mathbf{U}\|_0^2 + \lambda\|tr\mathbf{U}\|_0^2 + \|\mathbf{A}^{1/2}\nabla\mathbf{u}\|_0^2). \tag{198}$$

Again sufficient regularity of $\mathbf{U}$ and $\mathbf{u}$ is required for these estimates.

The efficient solution of these equations with standard multigrid algorithms depends on the weak coupling between the different parts of the equations (cf. subsection 3.2.2). But in the estimates shown above, the term $tr\mathbf{U}$ depends on λ and hence becomes dominant as $\lambda \to \infty$ (this happens when the material parameters approach incompressibility). As $tr\mathbf{U} = u_1 + u_4$ (with $\mathbf{U} = (u_1, u_2, u_3, u_4)^T$) this implies a strong coupling between these unknowns and hence will destroy the diagonal dominance of the system. To overcome this problem a simple rotation $\mathbf{Q}$ is applied to $\mathbf{U}$, which does not impose major problems as the displacements have to be recovered in the second step anyway (cf. [30], [32]).

For the formulation using the rotated set of unknowns $\mathbf{V} = (v_1, v_2, v_3, v_4)^T$, the following estimate holds:

$$c_8(\|\mathbf{V}\|_1^2 + \lambda^2\|\nabla v_1\| \leq \mathcal{J}_0(\mathbf{QV}) \leq c_9(\|\mathbf{V}\|_1^2 + \lambda^2\|\nabla v_1\| \tag{199}$$

Here the unknowns are clearly better decoupled, as only the gradients of v_1 appear in the L_2-norm.

Numerical results for these formulations were published in [30] and in [105]. Both publications were able to show that this method is able to handle even nearly incompressible material ($\nu \approx 0.5$) without convergence problems. In [30] also the performance of various multigrid schemes was examined. The authors were able to confirm numerically convergence rates of the multigrid scheme which were independent of h and the Lamé parameter λ.

5.3 Displacement-Stress Formulation

Looking at the basis of the equations of linear elasticity, they stem from the following equation:

$$\nabla \cdot \sigma = -\mathbf{f} \quad \text{in} \quad \Omega. \tag{200}$$

In this equation σ denotes the Cauchy stress tensor. Inserting the following constitutive equation, which is a linear relation between the stresses σ and the simplified Green strain tensor $\mathbf{E}$:

$$\sigma = \mathbb{C}\mathbf{E}(\mathbf{u}) \tag{201}$$

into Eq. (200) gives the well known equations of linear elasticity, Eq. (147). $\mathbb{C}$ denotes the elasticity tensor. Closely related is the compliance tensor $\mathbb{A}$, which is the inverse of $\mathbb{C}$ (cf. [35]):

$$\mathbf{E}(\mathbf{u}) = \mathbb{A}\sigma \tag{202}$$

As Eq. (200) is already first order it is a natural choice for a first order least squares formulation. The complete set of equations is then (cf. [29]):

$$\begin{aligned} \sigma - \mathbb{C}\mathbf{E}(\mathbf{u}) &= 0 \quad \text{in} \quad \Omega, & (203)\\ \nabla \cdot \sigma &= \mathbf{f} \quad \text{in} \quad \Omega. & (204) \end{aligned}$$

Alternatively Eq. (202) can be included and leads to (cf. [35]):

$$\begin{aligned} \mathbb{A}\sigma - \mathbf{E}(\mathbf{u}) &= 0 \quad \text{in} \quad \Omega, & (205)\\ \nabla \cdot \sigma &= \mathbf{f} \quad \text{in} \quad \Omega. & (206) \end{aligned}$$

As the elasticity tensor blows up for nearly incompressible material ($\nu \rightarrow 0.5$), the latter formulation is preferable in that case.

The following functionals were proposed for the use in a least squares finite element method from these equations (cf. [35] and [29]):

$$\mathcal{J}_1(\sigma, \mathbf{u}) = \|\nabla \cdot \sigma + \mathbf{f}\|_0^2 + \mu\|\mathbb{C}^{1/2}\sigma - \mathbb{C}^{1/2}\mathbf{E}(\mathbf{u})\|_0^2, \tag{207}$$

$$\mathcal{J}_2(\sigma, \mathbf{u}) = \|\nabla \cdot \sigma + \mathbf{f}\|_0^2 + \|\mathbb{A}\sigma - \mathbf{E}(\mathbf{u})\|_0^2, \tag{208}$$

and

$$\mathcal{J}_{-1}(\sigma, \mathbf{u}) = \|\nabla \cdot \sigma + \mathbf{f}\|_{-1}^2 + \|\mathbb{A}\sigma - \mathbf{E}(\mathbf{u})\|_0^2. \tag{209}$$

Actually the first two functionals $\mathcal{J}_1$ and $\mathcal{J}_2$ are very similar and correspond to the two different ways of formulating the relation between stresses and strains. Functional $\mathcal{J}_{-1}$ uses a negative norm to reduce the regularity demands on the right hands side from $\mathbf{f} \in L_2$ to $\mathbf{f} \in H^{-1}$, which is more natural in the context of linear elasticity as it can be shown that the weak form of the equation possesses a solution $\mathbf{u} \in H^1$ for $\mathbf{f} \in H^{-1}$ (cf. [43]).

For the bilinear form stemming from $\mathcal{J}_1$, coercitivity can then be shown in the product space $H_{\Gamma_N}(\mathrm{div}, \Omega)^2 \times H^1_{\Gamma_D}(\Omega)^2$ with the following scaled norm:

$$\|(\tau, \mathbf{v})\|_{M_1} = \left(\|\nabla \cdot \tau\|_{0,\Omega}^2 + \mu\|\mathbb{C}^{-1/2}\tau\|_{0,\Omega}^2 + \mu\|\mathbb{C}^{1/2}\mathbf{E}(\mathbf{v})\|_{0,\Omega}^2\right)^{1/2}. \tag{210}$$

For $\mathcal{J}_2$ the norm is:

$$\|(\tau, \mathbf{v})\|_{M_2} = \|\mathbf{E}(\mathbf{v})\|_0^2 + \|\tau\|_0^2 + \|\nabla \cdot \tau\|_0^2, \tag{211}$$

and for $\mathcal{J}_{-1}$:

$$\|(\tau, \mathbf{v})\|_{M_{-1}} = \|\mathbf{E}(\mathbf{v})\|_0^2 + \|\tau\|_0^2. \tag{212}$$

Using these norms, the following estimates can be established:

$$c_1\|(\tau, \mathbf{v})\|_{M_1} \leq \quad \mathcal{J}_1(\tau, \mathbf{v}) \quad \leq c_2\|(\tau, \mathbf{v})\|_{M_1} \tag{213}$$

$$c_3\|(\tau, \mathbf{v})\|_{M_2} \leq \quad \mathcal{J}_2(\tau, \mathbf{v}) \quad \leq c_4\|(\tau, \mathbf{v})\|_{M_2} \tag{214}$$

$$c_5\|(\tau, \mathbf{v})\|_{M_{-1}} \leq \quad \mathcal{J}_{-1}(\tau, \mathbf{v}) \quad \leq c_6\|(\tau, \mathbf{v})\|_{M_{-1}}. \tag{215}$$

Due to some specific properties (for details refer to [29]), the functional $\mathcal{J}_1$ needs special finite element approximations for the displacements in the incompressible limit. Therefore the use of nonconforming Crouzeix-Raviart elements or their quadratic counterpart, the elements from Fortin and Soulie (cf.[54]) are used to approximate the displacement. The slightly different functional $\mathcal{J}_2$ allows also the use of standard finite elements for the displacement, even in the incompressible limit. The stresses must be approximated by finite subspaces of $H(\mathrm{div})$ in both cases. Raviart-Thomas elements are used for this purpose.

Numerical tests in [29] confirm the result, that the convergence rates deteriorate in the incompressible limit, if conforming standard finite elements are used in the incompressible limit with functional $\mathcal{J}_1$. In contrast the functional $\mathcal{J}_2$ used in [35] gives optimal convergence rates in the incompressible limit even together with the standard conforming finite elements.

Both articles use the least squares functional as built-in error estimator and construct an adaptive algorithm, which is based on that estimator.

5.4 Summary

Several ways to transform the equations of linear elasticity into an equivalent first order system have been analysed in literature so far. They can roughly be divided into formulations, which are based on the compressible Stokes equations and other formulations.

While the vorticity formulation seems to be generally a good formulation for the Stokes equations, the vorticity might be inconvenient for structural descriptions. The stress formulation and the formulation using the displacement gradient seem to be closer to the requirements in structural problems. But the required weights in the stress formulation render it a problematic formulation, as the correct weight influences the solution properties. This could be acceptable in the case of homogeneous materials, but is probably difficult to handle if the material parameters also vary over the domain. Thus the negative norm and augmented variant of the formulation using the displacement gradient remain, together with the negative norm approach which directly solves the original second order problem without additional unknowns.

An interesting class of formulations stems from the two-stage approach, which is very similar to the augmented displacement gradient formulation of the Stokes equations. It is available in a pure L_2 variant and in a negative norm version in 2D and 3D. Furthermore it was demonstrated that it allows the use of efficient multigrid solution algorithms. A disadvantage of the L_2 approach can be seen in the higher regularity demands on the right hand side term and in the question how the conservation of rotational momentum should be weighted compared to the conservation of momentum.

The formulations, which introduce the physical stresses as new unknowns seem to work quite good. By using special element types, full H^1-coercitivity of the least squares functional is not required, which extends the solution space. But these element types are sometimes not easily available in standard finite element codes, which could be seen as a disadvantage of those methods. A clear advantage is the fact that the physical stresses are directly available as problem unknowns. Furthermore the built-in error estimator of the LSFEM seems to work quite good as it was demonstrated in [30].

6 Incompressible Navier-Stokes Equations

This section will start with a brief introduction to the Navier-Stokes equations. After that a short summary of the mathematical treatment of the nonlinearity will be provided. Finally the proposed least squares formulations for the Navier-Stokes equations will be shown in detail. These are basically the same formulations, which were already shown for the Stokes equations, only enriched by the nonlinear convective terms. The final section reports a few numerical results for the instationary Navier-Stokes equations. As these equations have not been analysed mathematically so far, no mathematical results can be presented.

6.1 Applications and Theory of the Navier-Stokes Equations

Fluid flows appear in many technical applications and are often relevant to the overall performance of technical constructions. A huge range of these applications is covered by the incompressible Navier-Stokes equations, which actually provide a simplified model of fluid flows. The simplifications are admissible as long as the square of the Mach number $Ma = q/c$, which expresses the relation between the peak velocity of the fluid q and the sonic speed of the medium c, satisfies $Ma^2 \ll 1$ (cf. page *xxii*, Vol.2, [57]).

The normal form of the Navier-Stokes equations reads:

$$-\nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = \mathbf{f} \quad \text{in} \quad \Omega, \tag{216}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in} \quad \Omega. \tag{217}$$

Here p denotes the normalised pressure $p = P/\rho$ with the fluid mass density ρ. ν is the kinematic viscosity (cf.[97]).

The solution of the nonlinear Navier-Stokes equations generally requires some kind of iterative scheme. In these iterative methods, the Oseen equations, which represent a linearisation of the Navier-Stokes equations, play an important role:

$$-\nu\Delta\mathbf{u} + \mathbf{u}_0\cdot\nabla\mathbf{u} + \nabla p = f \quad \text{in} \quad \Omega, \tag{218}$$

$$\nabla\cdot\mathbf{u} = 0 \quad \text{in} \quad \Omega. \tag{219}$$

Looking at the standard Galerkin methods for the Oseen equations, two problems appear. First the Oseen equations are a saddle point problem with the same structure as the Stokes equations (cf. subsection 4.1). Therefore either special element pairs satisfying the discrete LBB condition or stabilisation techniques are required. The second problem comes from the convective part. When the convection is dominant, the ellipticity of the equation degenerates and some stabilisation techniques like upwinding or SUPG must be applied to get a non oscillatory solution (cf. [85]).

As the LSFEM includes some kind of stabilisation, the convective parts do not require any special treatment. Therefore the finite element methods for the Oseen equations do not differ much from those for the Stokes equations. Differences are mainly limited to the preconditioning techniques. Also the mathematical analysis is similar to that of the Stokes equations.

6.2 Nonlinear Error Estimates

In the previous sections several error estimates were shown for least squares methods for linear problems. Similar estimates are required for the nonlinear Navier-Stokes equations. As these estimates were derived for least squares formulations, they are based on the theory for the solution of the nonlinear Navier-Stokes equations, which is developed in [55]. This theory is adapted to the nonlinear least squares functionals (cf. [19]).

The starting point are some theorems for general nonlinear problems of the following form (following section 1.6.3 in [19]):

$$F(\lambda, \varphi) = \varphi + T \cdot G(\lambda, \varphi) = 0 \tag{220}$$

Let $\mathbf{X}$ and $\mathbf{Y}$ be two Banach spaces and Λ be a compact interval in $\mathbb{R}$. T is a linear mapping $\mathbf{Y} \rightarrow \mathbf{X}$, and G is a C^2 map $\Lambda \times \mathbf{X} \rightarrow \mathbf{Y}$. A *branch of solutions* is a set $\{(\lambda, \varphi(\lambda)) | \lambda \in \Lambda\}$ with $F(\lambda, \varphi(\lambda)) = 0$ and the map $\lambda \rightarrow \varphi(\lambda)$ being a continuous function from Λ into $\mathbf{X}$. This branch is called *regular* if in addition the Fréchet derivative of F with respect to φ is an isomorphism from $\mathbf{X}$ to $\mathbf{X}$ for all $\lambda \in \Lambda$.

The discrete counterpart of Eq. (220) needs a subspace $\mathbf{X}_h \subset \mathbf{X}$ and an approximating operator $T_h : \mathbf{Y} \rightarrow \mathbf{X}_h$. Then the discrete version of Eq. (220) reads:

$$F_h(\lambda, \varphi_h) = \varphi_h + T_h \cdot G(\lambda, \varphi_h) = 0 \tag{221}$$

The convergence theorem needs two assumptions. First there must exist a Banach space $\mathbf{Z}$ which is continuously embedded in $\mathbf{Y}$ such that

$$D_\varphi G(\lambda, \varphi) \in \mathcal{L}(\mathbf{X}, \mathbf{Z}) \quad \forall \lambda \in \Lambda \quad \text{and} \quad \varphi \in \mathbf{X}. \tag{222}$$

Here D_φ denotes the Fréchet derivative of G with respect to φ. Furthermore the following approximation properties must be satisfied by the discrete operator T_h:

$$\lim_{h \rightarrow 0} \|(T_h - T)g\|_{\mathbf{X}} = 0 \quad \forall g \in \mathbf{Y} \tag{223}$$

$$\lim_{h \rightarrow 0} \|T_h - T\|_{\mathcal{L}(Z,X)} = 0 \tag{224}$$

The following theorem (taken from [19]) then guarantees convergence of the discretised nonlinear problem to the solution of the continuous nonlinear problem.

Theorem: 1 Let $\mathbf{X}$ and $\mathbf{Y}$ be Banach spaces and let Λ be a compact subset of $\mathbb{R}$. Assume that G is a C^2 mapping from $\Lambda \times \mathbf{X}$ into $\mathbf{Y}$ and that all second Fréchet derivatives of G are bounded on all bounded subsets of $\Lambda \times \mathbf{X}$. Assume that Eqs. (222–224) hold and that $\{(\lambda, \varphi(\lambda)) | \lambda \in \Lambda\}$ is a branch of regular solutions of Eq. (220). Then there exists a neighbourhood $\mathcal{O}$ of the origin in $\mathbf{X}$ and, for h sufficiently small, a unique C^2 function $\lambda \to \varphi_h \in \mathbf{X}_h$, such that $\{(\lambda, \varphi_h(\lambda)) | \lambda \in \Lambda\}$ is a branch of regular solutions of Eq. (221) and $\varphi_h(\lambda) - \varphi(\lambda) \in \mathcal{O}$ for all $\lambda \in \Lambda$. Moreover, there exists a constant $C > 0$ independent of h and λ, such that:

$$\|\varphi(\lambda) - \varphi_h(\lambda)\|_{\mathbf{X}} \leq C \|(T - T_h) \cdot G(\lambda, \varphi(\lambda))\|_{\mathbf{X}} \quad \forall \lambda \in \Lambda \tag{225}$$

A further theorem establishes error estimates in weaker norms. For this purpose two new Banach spaces $\mathbf{W}$ and $\mathbf{H}$ with

$$\mathbf{W} \subset \mathbf{X} \subset \mathbf{H} \tag{226}$$

are required. Furthermore some new assumptions have to be introduced. For all $\varphi \in \mathbf{W}$, the operator $D_\varphi G(\lambda, \varphi)$ may be extended as a linear operator of $L(\mathbf{H}, \mathbf{Y})$, with the mapping $\varphi \to D_\varphi G(\lambda, \varphi)$ being continuous from $\mathbf{W}$ into $L(\mathbf{H}, \mathbf{Y})$. And

$$\lim_{h \to 0} \|T_h - T\|_{L(\mathbf{Y}, \mathbf{H})} = 0 \tag{227}$$

With these assumptions the following theorem, which gives an error estimate in the norm of H, can be established.

Theorem: 2 Assume the hypotheses of Theorem 1, Eq. (226), Eq. (227) and in addition assume that

$$\forall \lambda \in \Lambda, \varphi(\lambda) \in \mathbf{W} \quad \text{and the function} \quad \lambda \to \varphi(\lambda) \in C^0(\Lambda, \mathbf{W}) \tag{228}$$

$$\forall \lambda \in \Lambda, D_\varphi F(\lambda, \varphi(\lambda)) \quad \text{is an isomorphism of} \quad \mathbf{H} \tag{229}$$

Then, for $h \leq h_1$ small enough, there exists a constant $K' > 0$, independent of h and λ, such that:

$$\|\varphi(\lambda) - \varphi_h(\lambda)\|_{\mathbf{H}} \leq K' \left(\|(T - T_h) \cdot G(\lambda, \varphi(\lambda))\|_{\mathbf{H}} + \|\varphi(\lambda) - \varphi_h(\lambda)\|_{\mathbf{X}}^2 \right) \quad \forall \lambda \in \Lambda \tag{230}$$

Now the main issue in the analysis of the least squares finite element methods for the Navier-Stokes equations is to cast the least squares formulation into the framework described above and to show that the hypotheses Eqs. (222–227) hold.

6.3 Velocity-Pressure-Vorticity Formulation

As it was shown in the previous section, the convergence of a numerical method for the nonlinear Navier-Stokes equations is closely related to the convergence properties of the numerical scheme for the linear Stokes equations.

Two ways could be considered to include the nonlinear convective terms into the first order formulation using the vorticity. Obviously it can be expressed with the velocity derivatives, which leads to:

$$\nu\nabla\times\omega+\mathbf{u}\nabla\mathbf{u}+\nabla p=\mathbf{f}\quad\text{in}\quad\Omega. \tag{231}$$

for the first part of the Navier-Stokes equations. Alternatively the vorticity can be used:

$$\nu\nabla\times\omega+\omega\times\mathbf{u}+\nabla p=\mathbf{f}\quad. \tag{232}$$

This first order formulation of the Navier-Stokes equations is probably one of the oldest and most used. It is considered in [92], [91], [93], [62], [19], [15] and many other publications. One advantage is the low number of additional unknowns (one for 2D problems and three for 3D problems) and the physical meaning of the vorticity, which is introduced as a new unknown.

Using the pure L_2 minimisation for this formulation leads to suboptimal convergence rates, if pure velocity boundary conditions are applied. This was first pointed out by Bochev in his PhD-Thesis (cf. [19] and related papers (cf. [15]). A prior analysis of the convergence, performed by Chang et.al. (cf. [40]), was shown to be wrong.

In his book Jiang shows an impressive number of examples, which use the velocity-vorticity formulation (cf. [62]). During his demonstrations, he emphasises two main aspects which are important for good performance. The first one is the use of reduced integration because the LSFEM leads to an overdetermined system of equations according to his opinion. One drawback is the fact he does not explain how to get the optimal number of quadrature points. Using 1 point quadrature throughout the complete domain can lead to an underdetermined system. Another unwanted property of his proposed underintegration scheme is the existence of spurious modes, which lead to oscillations in all variables. The workaround is to set the values at the nodes to the interpolated values between the integration points. Although he shows numerical convergence results for the Stokes equations his explanation for the observed behaviour is not completely satisfying.

He presents results for the driven cavity problem in 2D at $Re=10000$, the driven cavity in 3D at $Re=1000$ and a backward facing step in 3D at $Re=800$. By replacing the time derivative through a finite difference he shows how to solve instationary flow problems. As an example he presents the vortex shedding behind a cylinder. In most examples he gives a brief comparison with experimental or other numerical results and is able to show good agreement.

Tang et.al. used the LSFEM for the simulation instationary flows, often introducing some additional "difficulties". In [92] they solved the Navier-Stokes equations coupled with the thermal equations to simulate thermal driven flows. But they also verified the results for other benchmark problems like the lid driven cavity and the flow over an obstacle. Their results seem promising, but according to Gresho (Chapter 3.16.9, [57]) Tang admitted in a personal communication with Gresho that the method failed to get the right Strouhal number for a flow around a cylinder. According to Gresho the dissipative Euler backward method used by Tang could be responsible for that behaviour. In later articles Tang used the LSFEM in conjunction with the Crank Nicholson scheme for time integration, which is also the way proposed by Jiang for time accurate solutions. With this method Tang simulated a 3D lid driven cavity ([91]) and again thermocapillary flows in 2D and 3D ([93]).

A closely related method introduces the Bernoulli or total pressure $b = p + (u^2 + v^2 + w^2)$ (with $\mathbf{u} = (u, v, w)^T$). From the mathematical point of view the properties of the method do not change. The advantage of this formulation is the fact that a large part of the nonlinearity is absorbed by the definition of the total pressure. Hence the method has no derivatives involved in the nonlinear part. But Jiang states that numerical tests revealed slightly slow convergence ([65]).

6.4 Velocity-Velocity Flux-Pressure Formulation

In [12] Bochev et.al. analyse the augmented velocity-velocity-flux formulation for the Navier-Stokes equations. The main focus of that paper lies in the proof of a unique solution and convergence of the proposed Newton method. With the H^1-coercitivity the resulting variational principle for the first order system is equivalent to a system of coupled Poisson like equations. Hence the use of multigrid methods is straightforward and should lead to very fast numerical schemes. One major drawback of that method are the high regularity demands ($\mathbf{u} \in H^2$) on the solution, which might be too strong for the nonlinear Navier-Stokes equations.

Therefore the first order formulation is also examined in the negative norm variant ([13] and [18]), which makes the additional equations superfluous and reduces the regularity demands on the solution to $\mathbf{u} \in H^1$. In these articles Bochev et.al. show that the high regularity demands of the augmented method can really lead to inferior convergence in the case of discontinuous boundary conditions. For the efficient solution of the resulting algebraic system they propose a preconditioner which is performs very well in the numerical examples presented in the paper.

6.5 Displacement-Stress-Pressure Formulation

This formulation was used to compute approximate solutions for the Navier-Stokes equations in [50], [99], [7] and probably some more publications. A space-time version, including high order ansatz-functions, of this formulation was examined in [8].

The use of this formulation in conjunction with the p-Method has been proposed and analysed by Winterscheidt and Surana in [99]. They emphasise their treatment of the nonlinearity appearing in the Navier-Stokes equations. Instead of linearising the Navier-Stokes equations before applying the variational principle, they derive the variational formulation for the nonlinear equations and do then the linearisation. But they don't compare their way of linearising the Navier Stokes equations with the other way. Thus it must be considered as an hypothesis that their linearisation has any benefits over the other way.

In [7] also the influence of the applied weighting is examined. All unknowns are made dimensionless in their formulation (this is done in [99] as well). Two different weights are proposed in that article. Depending on the scaling applied to the stresses, the Reynolds number appears either in the momentum equations or in the stress equations. In the first alternative the viscous stress is chosen to scale the stresses. The second approach uses the characteristic kinetic energy to scale the stresses. Computationally the first choice is advantageous according to [7]. As it was already noted previously, the LSFEM tries to satisfy all equations at a time. How good these equations will be satisfied depends on the used weights for the different parts of the equation. Now the scaling using the characteristic viscous stress leads to a unbalance between different parts of the equations, if the Reynolds number increases. This is not the case, if the kinetic energy scale is used. A general recommendation in this paper is to chose the non dimensional form of the equations such that the scaling of the different parts of the equations does not change. Although this sounds quite reasonable this statement is in contrast to the mathematical convergence results, which demand a mesh dependent scaling to achieve optimal convergence rates.

In their publication they state that the h-version of the LSFEM is known to be inaccurate and hence they propose the p-version of the LSFEM for the Navier-Stokes equations. The reason might be they did not know the results which were obtained in the same year by Bochev ([14]), where he shows that a mesh dependent weighting parameter is required to achieve optimal accuracy for the Velocity-Stress-Pressure formulation of the Stokes equations. Although this is pure speculation, Winterscheid and Surana do not use any mesh dependent weighting in their formulation and ([14]) does not appear in their literature list. At least the statement that the h-method is inaccurate is only correct for the Stokes equations, when the pure L_2 minimisation is applied.

6.6 Summary

All least-squares formulations, which work for the Stokes equations and include the velocities as unknowns, could be used for the nonlinear Navier-Stokes equations. These are the weighted velocity-vorticity-pressure formulation, the weighted velocity-stress-pressure formulation and the negative norm and augmented velocity-velocity flux-pressure formulation. All of these formulations have been tested numerically. A rigorous mathematical analysis of the nonlinear problem exists for the velocity-vorticity-pressure formulation and for the velocity-velocity flux-pressure formulation.

Regarding the least squares methods for solving the Navier-Stokes equations for instationary flows, the situation is different. Only a few numerical tests have been performed so far. The velocity-vorticity-pressure formulation was tested with an Euler backward and Crank-Nicholson time discretisation and with a space-time approach. For the velocity-stress-pressure formulation a space-time approach was tested. No results exist for the velocity-velocity flux-pressure formulation.

For none of the formulations for the time dependent problem a rigorous mathematical analysis was done yet.

7 Applications of the LSFEM

Although several papers are concerned with the Stokes and Navier-Stokes equations, which were shown in the previous sections, the LSFEM was used successful in many other areas. The following sections should give a short and probably incomplete review of material, which exists for structural mechanics, computational fluid dynamics including non-Newtonian fluids, fluid-structure interaction problems, computational electromagnetics, the neutron transport equation and the biharmonic equation. For some of these areas more detailed information can be found in [62] which provides a broader overview about some of these topics. The structural and fluid mechanics parts are kept short as these fields were already discussed in detail in the previous parts.

7.1 Structural Mechanics

As least squares methods for the equations of linear elasticity were already shown in section 5 the discussion in this subsection will be limited to least squares methods for other areas of structural problems.

Thin structures appear in many engineering areas, like crash simulation or sheet metal forming. Shell elements provide the basis for accurate and efficient simulations. The Mindlin-Reissner theory and the Kirchhoff-Love theory provide the theoretical foundation for these shell elements (cf.[9]). The equations for the Mindlin-Reissner plate are examined in [28], [36] and [22].

These equations are difficult to handle with the classical Galerkin approach when the structures become very thin as the elements tend to lock. Several methods like stabilisation and multistage algorithms were developed to circumvent these problems. Nevertheless, according to [22] a completely satisfactory algorithm was not found yet.

The least squares approaches we found in literature are all based on a three-stage algorithm for the Mindlin-Reissner plate equations, which uses a Helmholtz decomposition and was proposed by Brezzi and Fortin in [25]. This approach decouples the Mindlin-Reissner equations into two simple elliptic equations and a saddle point problem, which can be seen as a perturbed Stokes equation. As the two elliptic problems are rather simple, the articles focus on the treatment of the saddle point problem.

In [22] a negative norm approach is used. For this approach optimal convergence results are presented and also an optimal multigrid algorithm is presented. The approach does not convert the saddle point problem into a first order system, but solves it directly by some sophisticated techniques. Both articles [36] and [28] transform the system into a first order system by introducing the gradient of the primary unknown. Therefore it is closely related to [67] and [32]. For the resulting first order system two different functionals are proposed. One uses a negative norm and provides optimal convergence and multigrid results under standard regularity assumptions, while the proposed pure

L_2 functional offers these properties only under enhanced regularity assumptions.

7.2 Computational Fluid Dynamics

Besides the important incompressible Navier-Stokes equations several other physical models exist for the description of fluid flows. Following roughly the classification used in [62] the first important differentiation is between viscous and inviscid flows. Assuming the flow field to be irrotational allows a further model simplification. The next distinction is between compressible and incompressible fluid models. Compressible fluid models become numerically more challenging if the fluid velocity approaches the sonic speed because shocks can develop and lead to numerical difficulties. Thus Jiang [62] differentiates between high and low speed compressible flows.

The most simple fluid model is probably that of an incompressible inviscid irrotational fluid:

$$\nabla \cdot \mathbf{u} = 0 \quad \text{in} \quad \Omega \tag{233}$$

$$\nabla \times \mathbf{u} = 0 \quad \text{in} \quad \Omega. \tag{234}$$

These equations and those of compressible inviscid irrotational flows were solved with the LSFEM in [63]. In his book Jiang emphasises the advantage, that the velocity field is directly available with full accuracy and has not to be recovered as in the FEM formulations which use the potential.

Far more important for technical applications are the Euler equations, which describe an inviscid fluid. The instationary incompressible Euler equations are solved using a modified velocity-vorticity-pressure formulation in [102]. Several benchmark problems like a standing vortex and a propagating vortex are used to verify this approach. In [62] the main results are presented as well.

In contrast to the Galerkin method, the shocks, which could appear in solutions of the compressible Euler equations, do not need special stabilisation techniques as the LSFEM already includes sufficient numerical diffusion. This on the other hand prevents a sharp resolution of shocks unless the grid along these shocks is very fine. An adaptive scheme is proposed as a way to circumvent these difficulties in [90]. In an iterative process the nodes of the finite element grid are concentrated in the areas with high errors. The presented results look quite promising and can be found also in [62]. In [74] a slightly modified least squares approach is used to compute the stationary solution of the Euler equations with a time stepping method. The numerical method presented in that article is conservative.

In [62] further results for more exotic fluid problems are presented. Examples include thermally driven fluid flows (cf. [106], [107]) and two fluid flows, where the least squares formulation for the incompressible Navier-Stokes equations is coupled with a Volume of Fluid (VOF) approach for interface tracking (cf. [104]).

7.3 Non-Newtonian Fluids

Especially the Velocity-Stress formulation is very attractive for the use in conjunction with non-Newtonian fluids because the stresses are available as unknowns. Publications from Bell and Surana ([7]) and Edgar and Surana ([50] analyse this application of the LSFEM.

The governing equations in both papers are modified versions of the Navier-Stokes equations based on the idea presented in Eq. (170) and Eq. (172). In [50] the equations of the axisymmetric case are examined, while [7] uses the two-dimensional form in Cartesian coordinates. The constitutive equation for the fluid is a power law, which sets the fluid viscosity into relation to the second invariant of the strain rate tensor and leads to a nonlinear variant of Eq. (172). Depending on the model parameter this leads to either shear thinning or shear thickening fluid behaviour.

In [7] this model is enriched by the energy equation, which models the heat transfer inside the fluid. Two interdependencies between the fluid and the temperature field are modelled. The viscosity depends on the temperature and the heat is generated by viscous dissipation.

For the numerical solution of the system of equations the p-Version LSFEM with pure L_2 minimisation is used. From the findings in [16] it becomes clear, why the p-Version is superior compared to the h-Version. Without mesh depended weighting the h-Version will not converge with optimal rates. Although this has not been analysed mathematically for the equations stemming from non-Newtonian fluids, the mathematical properties will probably not improve by introducing an additional difficulty like a velocity dependent stress-strain relation.

Both papers present a shear flow, a driven cavity and a sudden expansion as numerical examples for the non-Newtonian fluid. A shear flow and a sudden contraction are the test cases for the nonisothermal coupled model. For higher polynomial degrees the results are in good agreement with analytical solutions or benchmark problems. Nevertheless it must be considered more or less a heuristic approach, which seems to work through the general robustness of the LSFEM. From the mathematical point of view it is not clear, if that approach works or converges under all circumstances.

7.4 Fluid-Structure Interaction

Fluid-Structure phenomena appear in many technical and biological fields. In civil engineering bridges and high buildings could start to oscillate under windy conditions, if they are not designed carefully. Similar problems appear at the wings of planes which could start to flutter, which normally causes severe damage. In biology and medicine there is also a growing interest in these phenomena, as biological structures like arteries or other sorts of tissue can normally not be assumed to be rigid.

Currently the numerical solution of large scale fluid-structure interaction problems is mostly done by partitioned approaches, which couple existing codes for the different subproblems in a suitable way. The highly optimised codes for the subproblems often lead to fast coupled computations despite their slight theoretical disadvantages.

As pointed out in section 5 and 6, several least squares formulations for the Navier-Stokes equations and the equations of linear elasticity exist and were shown to be convergent. Therefore the idea to use the LSFEM for fluid-structure problems is not far fetched.

A first attempt in that direction considers stationary fluid structure problems and uses a mixedformulation in the sense that the numerical treatment of the structural part is performed by a commercial FEM code (ABAQUS) which uses a standard Galerkin formulation, while the equations describing the fluid are solved with the least squares finite element method (cf. [73]). The least squares formulation for the fluid is the well known velocity-vorticity-pressure formulation, which was discussed in detail in subsection 4.2. Coupling is achieved by a partitioned approach, where the two subproblems are solved successively.

In [66] a strongly coupled approach is proposed, which uses space-time finite elements for the solution of both subproblems. The fluid-formulation is also based on the velocity-vorticity-pressure formulation, while a standard space-time Galerkin formulation is used for the structural part. Due to the coupling terms, one big advantage of the LSFEM, the resulting symmetric matrix, is lost. Therefore the direct solver used in [66] is probably not well suited for really large problems.

The department of applied mathematics at the University of Boulder is also very active in the area of least squares methods for Fluid-Structure problems. The practical problems considered by them are fluid structure interactions in the human eye ([46],[60]) and the blood flow in arteries ([59]). Their approach is based on a simultaneous solution of the Navier-Stokes equations and the equations of linear elasticity. For the Navier-Stokes equations the stabilised velocity-velocity flux-pressure formulation (cf. subsection 4.4) is used and the structural part is discretised using the very similar displacement-displacement gradient formulation (cf. subsection 5.2). The use of the LSFEM for both subproblems allows the formulation of the coupling conditions as a least squares functional as well. As it is still a minimisation problem, the resulting system of equations will still be symmetric in contrast to the formulation used in [66]. Another interesting detail is the inclusion of the grid deformation into the coupled system of equations.

The least squares functionals for all three equations are fully H^1-coercive and can be solved very efficiently using multigrid preconditioning techniques. Hence in [60] the computational efficiency of an algebraic multigrid is solver is emphasised. In [59] the idea is used for instationary problems. Unfortunately that article does not provide any details how the numerical implementation is done.

7.5 Electromagnetics

In many engineering and electrical engineering disciplines there is a need to predict electromagnetic fields. The efforts to develop numerical methods for this purpose can be summarised under *computational electromagnetics* (CEM). The Maxwell equations, which were discovered more than hundred years ago, are the basis for these numerical methods.

A very detailed discussion of the numerical methods for these equations can be found in [62]. Originally the Maxwell equations are eight first-order equations with six unknowns. Therefore many authors state that the Maxwell equations are overdetermined. In his book Jiang [62] tries to show that this assertion is not true. The equations which ensure that the electric and magnetic field intensities are divergence free are left out by many authors, because it can be shown that an originally divergence free field keeps this property. Jiang makes this sloppy treatment responsible for problems with spurious solutions, which appear in many proposed numerical schemes. Instead he adds two slack unknowns, which are exactly zero, to show that the Maxwell equations are well defined.

Using a simple theorem, which states that a function $\mathbf{u}$ satisfying $\nabla \times \mathbf{u} = 0$ and $\nabla \cdot \mathbf{u} = 0$ is exactly zero if some special boundary conditions are satisfied, the time harmonic Maxwell equations can be transformed into the Helmholtz equations, which present an equivalent second order system under some conditions. Due to the second order operators, these equations are more convenient for the treatment with the Galerkin FEM.

Jiang [62] presents several examples for electrostatic and magnetostatic computations. In these cases, the Maxwell equations can be simplified significantly and actually reduce to simple div-curl systems, which can be solved efficiently by the LSFEM. For the solution of time-harmonic fields Jiang uses again the first order Maxwell equations as a basis for his least squares formulation. The numerical results include a split cylinder with different complex permittivity and an off-centre cylinder which is embedded into a larger cylinder.

The final section about CEM in [62] deals with the numerical simulation of transient wave scattering problems and is based on the papers [101] and [103]. A Crank-Nicholson scheme is used for time discretisation of the first order Maxwell equations. The results for the radar cross section (RCS) are compared with the analytical solution of a simple test case, which consists of a cylinder embedded into a larger cylinder. The numerical results, which are obtained with the LSFEM are in good agreement with the analytical solution.

An article written by Bao and Yang (cf. [6]) examines the mathematical properties of a least squares approach for time harmonic wave scattering problems. The focus in that article is mainly on the treatment of the conditions on interfaces, which are inclu-

ded into the least squares functional in some broken norms (i.e. $H^{1/2}$). Two different functionals are derived for the TM version and the TE version of the Maxwell equations. For both cases coercitivity results and optimal error estimates are derived.

An approach for the same type of problem (time harmonic wave scattering) based on the second order Helmholtz equations is presented in [70] and [71]. Actually the formulation proposed in both articles is mainly the same. In [71] a negative norm approach is added as a variation of the pure L_2 functional and used mainly to prove the coercitivity results, while the L_2 functional is used for the computations. The coercitivity results are not uniform in the wavenumber k, which leads to problems with standard multigrid schemes using standard smoothing schemes like Gauss Seidel or Jacobi. To circumvent these difficulties, a special multigrid method is proposed, which uses exponential interpolation and multiple coarsening (cf. [23]). Some numerical examples show, that this non-standard multigrid scheme achieves much better convergence rates than a standard multigrid scheme for this type of problem.

7.6 Neutron Transport Equation

The neutron transport equation is a linearised version of the Boltzmann equation based on neutron conservation. As the name suggests, the neutron transport equation describes the distribution of neutrons in some media (cf. [1]). From a numerical point of view it exhibits several difficulties, as the unknown variable is the *angular flux* which depends not only on the spatial coordinates but also on the angle. The behaviour of the solution depends on the total- and absorption cross sections. Depending on the ratio of these values, the transport or diffusion processes can become dominant and influence the solution. Numerically this behaviour is quite challenging.

Another difficulty lies in the basis functions for the angular directions. Standard finite element functions are not well suited for this purpose because they can lead to *ray effects*. These can be avoided by using Legendre polynomials (spherical harmonics). But these basis functions also have some disadvantages as the treatment of the boundary conditions becomes quite difficult (cf. [78]).

In [86] the application of a least squares finite element method for these equations was examined with a special focus on the solution in diffusive regimes. Using an intelligent scaling of the different parts of the least squares functional Manteuffel and Ressel are able to show optimal convergence properties of the least squares formulation in [78]. Furthermore they propose a multigrid algorithm, which obtains very good convergence rates (cf. [78]).

While in [78] not much attention was payed to the boundary conditions, these are analysed in more detail in [79]. Here the admissible boundary conditions are derived. Additionally error estimates are established for the angular basis functions.

The main advantage of the least squares approach for the Neutron transport equation is the generality of the proposed approach according to [78]. Another feature, which makes the least squares formulation attractive is again its built-in error estimator (cf. [78]), which could be used for adaptive schemes and would lead to a very efficient numerical scheme in conjunction with the proposed multigrid solver.

A closely related application of the LSFEM can be found in [26]. Here the Boltzmann equations, which have a very similar structure, are solved with a least squares method. Again the focus of that article lies on a special multigrid scheme, which is tailored for the problem. In several numerical tests, the authors show that the proposed scheme performs better than standard multigrid schemes, although it still does not reach the optimal complexity $O(n)$ in all cases.

7.7 Biharmonic Equation

In the previous sections several examples of second order partial differential equations were presented together with the corresponding least squares functionals. To show that the LSFEM is not limited to first or second order equations, the last area of applications presented in this report will be about some publications from Thatcher, who considers least squares methods for the fourth order biharmonic equation (cf. [95]).

The biharmonic equation is:

$$\nabla^4 u = q \quad \text{in} \quad \Omega. \tag{235}$$

This equation can be transformed into an equivalent system of two second order equations by introducing variables φ and ψ. In a next step these two second order equations can be further transformed into a first order system. The four unknowns in that system are the spatial derivatives of φ and ψ. Hence a recovery procedure to obtain u will be necessary. For this first order formulation Thatcher shows H^1-coercitivity and optimal convergence rates.

Actually the interest for this biharmonic equation comes from a first order formulation for the Stokes equations, which is equivalent to the first order system coming from the biharmonic equation (cf. [94]). Using the basic conservation equations (conservation of mass and momentum) and introducing parts of the Cauchy stress tensor and the derivatives of the stream function as new unknowns leads to a first order system, which is equivalent to the system coming from the biharmonic equations up to some scaling factors.

8 Solution Methods for the LSFEM Discretisation

Especially for large problems which often appear in computational fluid dynamics, the existence of efficient solution methods for a proposed numerical scheme is a must. While for two dimensional problems the use of a direct solver often allows one to circumvent this topic, for three dimensional problems with a large number of elements, iterative methods are currently the only known way to solve the resulting systems of equations. In this section direct solvers will not be considered because their algorithm is normally completely independent of the used discretisation except for symmetric problems which allow slightly more efficient algorithms.

8.1 Iterative Methods

Most known iterative methods can roughly be divided into two groups (beside multigrid or -level methods), simple fixpoint iteration schemes and Krylov subspace methods. Nowadays the fixpoint iteration schemes are most often used as preconditioners for Krylov schemes or as smoothers for multigrid methods.

The simplest and also most robust Krylov subspace method is the conjugate gradient method. Its only disadvantage is that it is limited to symmetric problems. Other algorithms like Bicgstab or GMRES extend the ideas of the conjugate gradient method to nonsymmetric problems but are less robust. The performance of all Krylov subspace method depends crucially on efficient preconditioning, because the number of iterations depends on the condition number of the system of equations.

Because the LSFEM is basically a minimisation, the resulting system of equations is always symmetric and positive definite. Therefore an algorithm which will always work is the preconditioned conjugate gradient method. This can be seen as one of the strengths of the LSFEM because it allows the implementation of black-box solvers where the same code framework can solve arbitrary partial differential equations.

B.Jiang [62] proposes this approach using a simple Jacobi preconditioner and a matrix-free conjugate gradient implementation. Together with his favoured reduced integration this seems to produce acceptable efficiency and is probably the form of algorithm which allows the largest problems sizes with a given amount of memory. But Gresho reveals in his book ([57]) that in personal communications B.Jiang mentioned several thousand iterations for fluid mechanics problems of average size. So there might be more efficient algorithms, which take the special structure of the problem into account.

Actually discretisations with the first order systems least squares method will always lead to matrices with a condition number in the order of $O(h^{-2})$, where h characterises the element size. Therefore the conjugate gradient algorithm will converge in approximately $O(h-1)$ iterations, because the number of iterations is approximately

proportional to the square root of the condition number (cf. chap.2 in [85]).

8.2 Multigrid Methods

Multigrid methods are probably the most efficient algorithms available today for the solution of large systems of equations (cf. [58]). The basic idea is to introduce interpolation and restriction operators which transfer the residual to the finer or coarser discretisation respectively (cf. [58]). On the different levels smoothers, which are often simple iterative solvers like Gauss-Seidel or SOR, can optimally reduce the residuum. This is due to the fact that the smoothers have only a certain range of spatial frequencies which can be damped away efficiently (cf. sec. 2.2 in [58]). Ideal multigrid solvers achieve convergence rates, which are independent of the element size h and hence obtain a performance of $O(n)$ which means the work is only proportional to the number of unknowns. This is obviously the best result which can achieved on normal computers.

Not only that multigrid solvers give quasi-optimal performance on single processors, the linear complexity gives these algorithms a good scalability which allows efficient parallelisation up to a large number of processors.

Beside their overwhelming advantages there is one major drawback. One multigrid algorithm does not work for arbitrary problems with the same efficiency. For every problem in hand the corresponding multigrid algorithm has to be developed. And it is not clear if there exists a multigrid algorithm for all problems.

8.2.1 Multigrid for the Standard FEM

The first multigrid method was formulated by Fedorenko [52] for the Poisson equation on a square. Later convergence proofs for the FEM followed in [82]. For the discretisation of the Laplace operator these methods achieve an optimal performance and are actually capable of solving the Laplace equations with a computational effort being proportional to the number of unknowns. But it was quickly recognised that the performance degrades substantially, when the system is changed into an advection-diffusion system with a certain amount of convection (cf. [58]). Mathematically this correspond to the loss of ellipticity of the corresponding bilinear form (cf. [24]). Algebraically it can be seen from the effect of the standard smoothing operations (Gauss-Seidel or SOR), which become more and more ineffective with increasing convection (cf. chap.10.4.3 in [58]). To circumvent these difficulties, smoothing iterations which take the direction of the convection into account, were proposed. A generalisation of this approach are the incomplete LU decompositions (ILU), which provide an approximate LU decomposition of the resulting stiffness matrix. These ILU decompositions are quite reliable and work well as smoothers and preconditioners for several problems (cf. [85]), but are more expensive than Gauss-Seidel or SOR smoothers. The quest for

optimal multigrid algorithms is still an active field of research and many publications appear regularly about this topic.

8.2.2 Multigrid for the LSFEM

In the previous section the importance of full H^1-coercitivity was already mentioned. These coercitivity results have a direct impact onto the performance of multigrid schemes for least squares methods. Actually, the solution of the Laplace operator corresponds to minimising the H^1 norm of the unknown function under the given boundary conditions. If the least squares functional is H^1-coercive, this implies that the least squares functional is spectrally equivalent to the Laplace operator of the unknown function. This means that the eigenvalue distribution, which is one relevant factor for the multigrid performance, is similar to that of the Laplace operator. Therefore H^1-coercitivity is one requirement to achieve optimal performance with standard multigrid schemes (i.e. normal prolongation and restriction operators and Gauss-Seidel or SOR smoothing) (cf. [12] and several other least squares publications).

Another issue, which has a strong influence on the performance of multigrid algorithms, is the coupling between the different unknown variables. This aspect is documented in [32], where the coupling of the unknowns becomes stronger in the incompressible limit and some special treatment is required. Also in [80] this issue is discussed for the convection diffusion equation. An exponential scaling of the different terms is proposed in that article to decouple the different unknowns.

A detailed analysis of least squares methods for linear hyperbolic first order formulations including multigrid methods for this kind of problem can be found in [83]. For the equations of linear elasticity an optimally convergent first order formulation and an efficient multigrid scheme were proposed in [32] and tested in [30]. For the Boltzmann equations and the Neutron transport equations multigrid schemes were proposed in [26] and [78]. The Helmholtz equations need a special multigrid algorithm, which was developed in [70].

For the Navier-Stokes equations it is not completely clear, if there exists an optimal geometric multigrid algorithm. Bochev proposed a multigrid scheme for the stabilised velocity-velocity flux-pressure formulation (cf. [12]), which is H^1-coercive. This formulation was used for the solution of Fluid-Structure interaction problems (cf. [60]) together with an algebraic multigrid method, which performed quite well. But other numerical results, which prove good multigrid performance for the Navier-Stokes equations in one of the proposed first order formulations did not appear in the literature so far.

8.2.3 Multigrid for LSFEM Formulations of Nonlinear Problems

Basically two ways exist to apply multigrid methods to nonlinear problem. The first way considers the linear subproblems which have to be solved in each inexact Newton iteration. As these subproblems are linear, the standard multigrid theory can be applied without modifications. An example for this algorithm can be found in [89], where the equations describing saturated subsurface flows are solved with least squares formulation and a multigrid algorithm.

The other approach includes the nonlinearity into the multigrid scheme and tries to solve the fully nonlinear problem on the different grids. Some authors state that the latter approach could lead to more efficient solution procedures, but requires more sophisticated analysis to achieve good performance (cf. [12]).

In [44] a method is proposed which starts on the coarsest level and computes a solution of the fully nonlinear problem on that level. This solution is then interpolated to the finer mesh, where one Newton iteration is performed. An algebraic multigrid method (AMG) is used to solve the linear subproblem coming from the Newton linearisation. The proposed scheme is used for the solution of elliptic grid generation (EGG) problems in [46] and [45].

A detailed comparison between the full nonlinear multigrid approach and the Newton approach using the multigrid scheme only for the linear subproblems can be found in [68]. The result is a bit astonishing, as it comes out that the full nonlinear multigrid approach is not the most efficient algorithm for the problem considered in that thesis.

9 Conclusion

In the previous chapters least squares finite element formulations for several applications were shown. The focus was mainly put onto linear problems, as these are the best examined so far in the least squares finite element literature.

Although the least squares finite element method is considered an universal approach for solving all problems appearing in the numerical treatment of partial differential equations by some authors, the presented results show that this is clearly not the case. But on the other hand the results show also that the least squares idea can lead to efficient and accurate solution methods for problems, which exhibit severe problems if solved with the standard Galerkin approach. But for success these methods have to be examined carefully with the mathematical tools developed for the analysis of finite element methods.

One of the most useful features is probably the symmetry and positive definiteness of the resulting system of equations. As already mentioned in section 8, this allows the use of the robust preconditioned conjugate gradient method. Multigrid methods would even be more efficient and could lead to numerical solvers with a linear complexity. In contrast to the multigrid methods for the Galerkin variational principle, which need often complex smoothers for good performance, the multigrid methods for the LSFEM normally need only standard smoothers like SOR or Gauss-Seidel. But although impressive results were achieved for some important problems, a geometric multigrid method is still missing for the Navier-Stokes equations.

Due to the built-in error estimator of the least squares finite element methods, several papers use adaptive schemes with great success. But the concept of dual error estimators (cf. [5] and the references therein), which refine the grid not with respect to the global error, but with respect to the error in some user specified functional, has not been used in conjunction with the LSFEM yet, according to our knowledge. The sharp error estimates for the LSFEM and the bilinear forms, which are always self adjoint by construction, should be very useful for dual error estimators.

Further open questions in the area of least squares finite element methods cover the solution of coupled problems. Some first publications have appeared about fluid-structure problems, and some previous works also considered problems like thermally driven flows, or thermocapillary flows. But several coupled problems have not yet been treated by the LSFEM. Especially when the coupling conditions are formulated in a least squares sense as well, this should lead to symmetric positive definite matrices, which should simplify the solution process.

Another area in the least squares finite element methods, which has not been examined, is the analysis of instationary problems. Currently two schemes exist for the solution instationary problems. The finite difference approach was examined for the convection equation, while the space-time finite element approach was analysed for

parabolic problems. But a theory for the instationary Stokes or Navier-Stokes equations is still missing. For the instationary equations of linear elasticity even a working first order formulation is missing.

Closely related to the last issue is the treatment of nonlinear structural problems. Up to now most papers only consider the equations of linear elasticity. But although the extension to geometric and material nonlinearities should be relatively straightforward, it has not been done yet.

The least squares finite element method is still a bit away from the mainstreamöf the finite element research. But the interest in this numerical method has been growing continuously the last years, which allows the conclusion that it probably has some interesting features. Despite the fact that substantial progress was made in the analysis of the least squares finite element methods for several partial differential equations, several open questions remain and provide space for interesting research.

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